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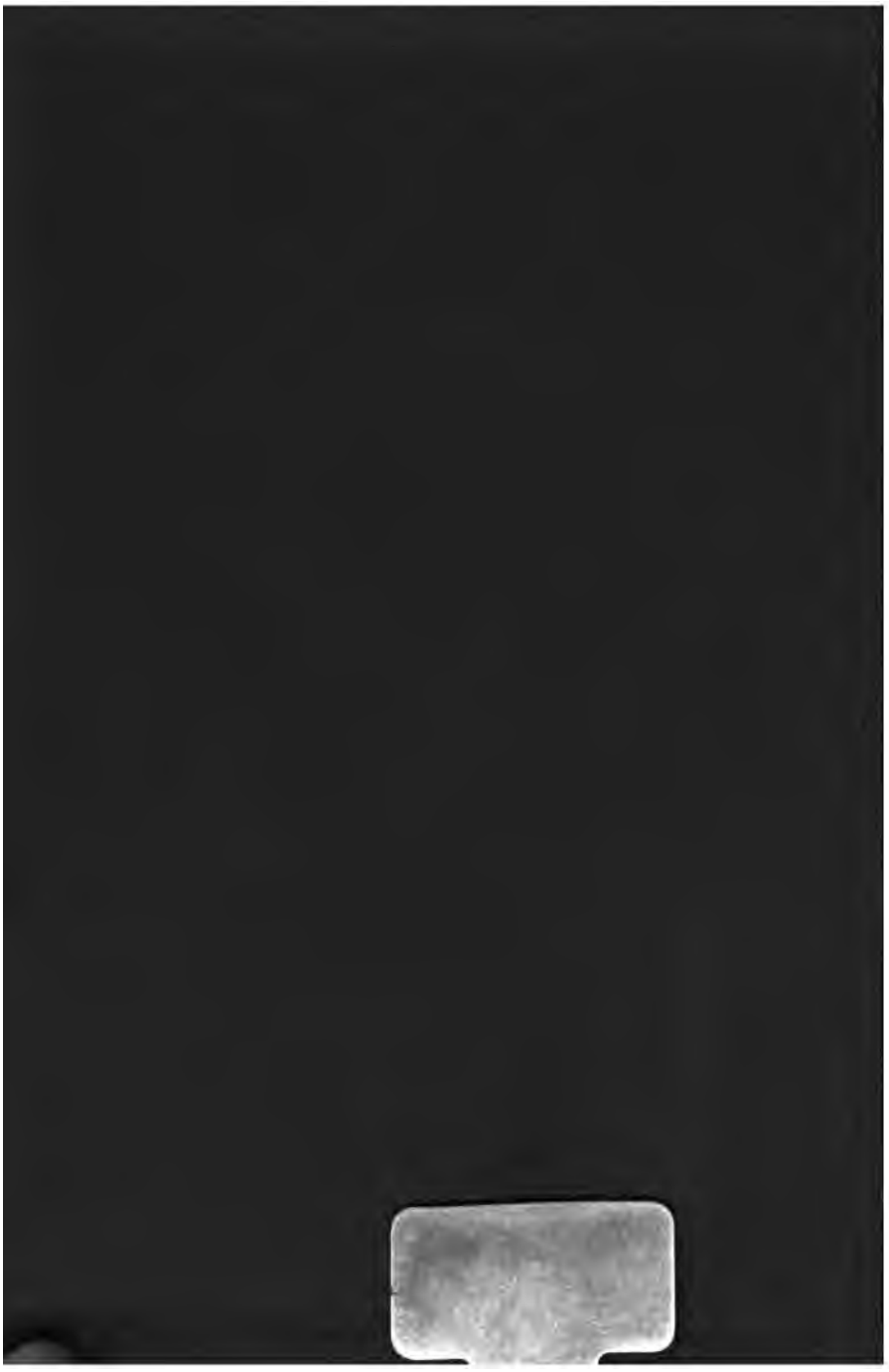

KEY

TO

COLENSO'S STUDENT'S ALGEBRA

LONDON : PRINTED BY
SPOTTISWOODE AND CO., NEW-STREET SQUARE
AND PARLIAMENT STREET

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COLENSO'S STUDENT'S ALGEBRA

BY THE

REV. J. HUNTER, M.A.

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KEY.

Ex. 1.

1. $6+10+12+12+6+2+0=48$.
2. $12+5-12+12-5+0=12$.
3. $15-24-24+21+4-0=-8$.
4. $-18+10+12-4+1=1$. 5. $30+100-24+0=106$.
6. $0-15+32-18=-1$. 7. $-90-48+80-120=-178$.
8. $150-192+360-168=150$. 9. $990-228+0-312=450$.
10. $360-240+72-0=192$.

Ex. 2.

1. $6+5+1-1=11$.
2. $3-3+1=1$.
3. $3-5+2=0$.
4. $1+18+75+0=94$.
5. $9+90-10+0=89$.
6. $1-15+75-125=-64$.
7. $1-12+54-108+81=16$.
8. $300-45+9=264$.
9. $1+5-1=5$.
10. $1-1+7-4=3$.

Ex. 3.

1. $2+10+21-8=25$.
2. $12-24+6-9=-15$.
3. $2+10-4+4=12$.
4. $1+6-4+3=6$.
5. $5+6+6+4=21$.
6. $10+9+8-5=22$.
7. $15+12-24+4=7$.
8. $5+6-2+4=13$.
9. $25-18+12-4=15$.
10. $6+30-36+4=4$.

Ex. 4.

1. $0+0+10+36=46$.
2. $6+4+16-2=24$
3. $4+15+16=35$.
4. $6+6-2=10$.

5. $30\times8\times30=7200$. 6. $9\times1\times3\times5=135=144-9$.
7. $2\times4=8=16-13+1+2\times2$.
8. $4+36+16+64=120=4\times30$.

Ex. 7.

1. $a-x-2x+a-2+2a+3-2x-1+x=4a-4x$.
2. $a^3-2a^2c+3ac^2-a^2c+2a^3-2ac^2+a^3-ac^2-a^2c$
$=4a^3-4a^2c$.
3. $2x^2-2y^2-z^2-3y^2-2x^2+z^2-3z^2+2y^2+x^2=x^2-3y^2-3z^2$.
4. $x^3+ax^2+a^2x-y^3+by^2-b^2y+z^3+cz^2+c^2z-x^3+y^3-z^3$
$+ax^2+by^2+cz^2-a^2x+b^2y-c^2z=2ax^2+2by^2+2cz^2$.
5. $a^2-b^2+c^2-b^2+c^2-a^2+c^2-b^2+a^2=a^2-3b^2+3c^2$.
6. $2a^2-3ab+b^2-a^2+4ab+b^2+2b^2-a^2+ab=2ab+4b^2$.
7. $x^3+y^3-3x^2y-3xy^2-x^3+3x^2y+3xy^2-y^3=0$.
8. $2x-3y+z-y-2x+z+3z-x+2y-2x+y-z$
$=-3x-y+4z$.
9. $1-1+1-4x+2x-3+5x-2+5x-4=8x-8$.
10. $2a-3b-c+2d-2a+3b-c+2d+2a-3b-c-2d-2a$
$+3b-c+2d=-4c+4d$.

Ex. 9.

2. $ax+x+ax-x-by+y-by-y=2(ax-by)$.
3. $(a+c+b-c)x^2+(3b-3a+2a+2b)xy+(b-c+a-b)y^2$
$=(a+b)x^2-(a-5b)xy+(a-c)y^2$.
4. (i) $(a+b+a-b)x+(b+c-b+c)y=2(ax+cy)$.
(ii) $(a+b-a+b)x+(b+c+b-c)y=2b(x+y)$.
5. (i) $(2a+2b-3a+3b)x+(3b+3c+2a-2c)y$
$=-(a-5b)x+(2a+3b+c)y$.
(ii) $(-2b-c+a-2b)x+(a-2b-b-2c)y$
$=(a-4b-c)x+(a-3b-2c)y$.
(iii) $(-a+5b+a-4b-c)x+(2a+3b+c+a-3b-2c)y$
$=(b-c)x+(3a-c)y$.

6. (i) $(2a+2b+3a-3b)x+(3b+3c-2a+2c)y$
$=(5a-b)x-(2a-3b-5c)y.$
(ii) $(-2b-c-a+2b)x+(a-2b+b+2c)y$
$=-(a+c)x+(a-b+2c)y.$
(iii) $(5a-b-a-c)x-(2a-3b-5c-a+b-2c)y$
$=(4a-b-c)x-(a-2b-7c)y.$

7. (i) $(2a+2b+2b+c)x+(3b+3c-a+2b)y$
$=(2a+4b+c)x-(a-5b-3c)y.$
(ii) $(-3a+3b-a+2b)x+(2a-2c+b+2c)y$
$=-(4a-5b)x+(2a+b)y.$
(iii) $(2a+4b+c-4a+5b)x-(a-5b-3c-2a-b)y$
$=-(2a-9b-c)x+(a+6b+3c)y.$

8. (i) $(2a+2b-a+2b)x+(3b+3c+b+2c)y$
$=(a+4b)x+(4b+5c)y.$
(ii) $(-3a+3b+2b+c)x+(2a-2c-a+2b)y$
$=-(3a-5b-c)x+(a+2b-2c)y.$
(iii) $(a+4b-3a+5b+c)x+(4b+5c+a+2b-2c)y$
$=-(2a-9b-c)x+(a+6b+3c)y.$

Ex. 10.

15.

x^2-ax+b
$x-c$
x^3-ax^2+bx
$-cx^2+acx-bc$
$x^3-(a+c)x^2+(ac+b)x-bc$

x^2-ax+b
x^2+ax-c
$x^4-ax^3+bx^2$
$+ax^3-a^2x^2+abx$
$-cx^2+acx-bc$
$x^4-(a^2-b+c)x^2+a(b+c)x-b$

16. $1-ax+bx^2-cx^3$
$1+x-x^2$
$1-ax+bx^2-cx^3$
$+x-ax^2+bx^3-cx^4$
$-x^2+ax^3-bx^4+cx^5$
$1-(a-1)x-(a-b+1)x^2+(a+b-c)x^3-(b+c)x^4+cx^5.$

18. $ax-by$
$ax+cy$

a^2x^2-abxy
$+acxy-bcy^2$

$a^2x^2-a(b-c)xy-bcy^2$
$ax-dy$

$a^3x^3-a^2(b-c)x^2y-abcxy^2$
$-a^2dx^2y+a(bd-cd)xy^2+bcdy^3$

$a^3x^3-a^2(b-c+d)x^2y-a(bc-bd+cd)xy^2+bcdy^2$

19. $2x-m$ $\quad$ $2x+n$
$x+2m$ $\quad$ $x-2n$

$2x^2-mx$ $\quad$ $2x^2+nx$
$+4mx-2m^2$ $\quad$ $-4nx-2n^2$

$2x^2+3mx-2m^2$ $\quad$ $2x^2-3nx-2n^2$
$2x^2-3nx-2n^3$

$4x^4+6mx^3-4m^2x^2$
$-6nx^3-9mnx^2+6m^2nx$
$-4n^2x^2-6mn^2x+4m^2n^2$

$4x^4+6(m-n)x^3-(4m^2+9mn+4n^2)x^2+6mn(m-n)x$
$+4m^2n^2$

20. x^2+ax-b^2
x^2+bx-a^2

$x^4+ax^3-b^2x^2$
$+bx^3+abx^2-b^3x$
$-a^2x^2-a^2x+a^2b^2$

$x^4+(a+b)x^3-(a^2-ab+b^2)x^2-(a^3+b^3)x+a^2b^2$
$x-(a+b)$

$x^5+(a+b)x^4-(a^2-ab+b^2)x^3-(a^3+b^3)x^2+a^2b^2x$
$-(a+b)x^4-(a^2+2ab+b^2)x^3+(a^3+b^3)x^2$

$x^5-\{2a^2+2b^2+ab\}x^3$
$+(a^4+a^3b+ab^3+b^4)x-a^2b^2(a+b)$

$+(a^4+a^3b+a^2b^2+ab^3+b^4)x-a^2b^2(a+b)$

Ex. 11.

5. $(x^2-a^2)\times(x^2-4a^2)=$&c.

6. $(m^2x^2-4n^2y^2)\times(m^2x^2-9n^2y^2)=$&c.

7. $3(a^2-4ax+4x^2)+2(a^2-4x^2)+(9x^2-a^2)$
$-(4a^2-12ax+9x^2)=$&c.

8. See (24): $\{(x^2+2y^2)+2xy\}\times\{(x^2+2y^2)-2xy\}$
$=(x^2+2y^2)^2-4x^2y^2=x^4+4x^2y^2+4y^4-4x^2y^2=$&c,
$\{(2a^2+b^2)-3ab\}\times\{(2a^2+b^2)+3ab\}=(2a^2+b^2)^2-9a^2b^2$
$=$&c.

9. $(a+b)^2-c^2=$&c, $(a+c)^2-b^2=$&c, $\{a+(b+c)\}$
$\times\{a-(b+c)\}=a^2-(b+c)^2=$&c.

10. $(a-b)^2-c^2=$&c, $\{c+(a-b)\}\times\{c-(a-b)\}$
$=c^2-(a-b)^2=$&c, $(c-b)^2-a^2=$&c.

11. $4a^2-(b-3c)^2=$&c, $b^2-(2a-3c)^2=$&c.

12. $4a^2-(b+3c)^2=$&c, $\{-3c+(2a-b)\}\times\{-3c-(2a-b)\}$
$=9c^2-(2a-b)^2=$&c.

13. $(a+c)^2-(b+d)^2=$&c, $(a+d)^2-(b+c)^2=$&c,
$(b+c)^2-(a+d)^2=$&c.

14. $(a+d)^2-(2b-3c)^2=$&c, $(3c+d)^2-(a-2b)^2=$&c,
$(a+3c)^2-(2b-d)^2=$&c.

Ex. 13.

1. $x-a]\ x^3-(a+p)x^2+(q+ap)x-aq\ [\ x^2-px+q$

$x^3-\ \ ax^2$

$-\ \ px^2+(q+ap)x$

$-\ \ px^2+\ \ apx$

$+\ \ qx-aq$

$+\ \ qx-aq$

2. $mz-n$]

$maz^3+(mb-na)z^2-(mc+nb)z+nc$ [az^2+bz-c

$maz^3-\ na\ z^2$

$+mb\,z^2-(mc+nb)z$

$+mb\,z^2-\ nb\ z$

$-\ mc\,z+nc$

$-\ mc\,z+nc$

3. $y\ -1$]

$y^5-my^4+ny^3-ny^2+my-1$ [$y^4+(1-m)y^3$

$y^5-\ y^4$ $+(1-m+n)y^2+(1-m)y+1$

$+(1-m)y^4+ny^3$

$+(1-m)y^4-(1-m)y^3$

$+(1-m+n)y^3-ny^2$

$+(1-m+n)y^3-(1-m+n)y^2$

$+(1-m)y^2+my$

$+(1-m)y^2-(1-m)y$

$+y-1$

$+y-1$

4. $a\ -(b-c+d)$]

$a^2-2ad-(b^2-2bc+c^2-d^2)$ [$a+(b-c-d)$

$a^2-a(b-c+d)$

$+a(b-c-d)-(b^2-2bc+c^2-d^2)$

$+a(b-c-d)-(b^2-2bc+c^2-d^2)$

5. $a\ -(b-3c)$]

$a^2+a(b+2c)-(2b^2-7bc+3c^2)$ [$a+(2b-c)$

$a^2-a(b-3c)$

$+a(2b-c)-(2b^2-7bc+3c^2)$

$+a(2b-c)-(2b^2-7bc+3c^2)$

6. $a-(b-c)]$
$a^3+3abc-(b^3-c^3)\ [\ a^2+a(b-c)+(b^2+bc+c^2)$
$a^3-a^2(b-c)$
$+a^2(b-c)+3abc$
$+a^2(b-c)-a(b^2-2bc+c^2)$
$+a(b^2+bc+c^2)-(b^3-c^3)$
$+a(b^2+bc+c^2)-(b^3-c^3)$
$a-(b+c)]$
$a^3-3abc-(b^3+c^3)\ [\ a^2+a(b+c)+(b^2-bc+c^2)$
$a^3-a^2(b+c)$
$+a^2(b+c)-3abc$
$+a^2(b+c)-a(b^2+2bc+c^2)$
$+a(b^2-bc+c^2)-(b^3+c^3)$
$+a(b^2-bc+c^2)-(b^3+c^3)$

7. Arranging in *descending* powers of x, we have
$x-(2y-1)]$
$x^3+6xy-(8y^3-1)\ [\ x^2+x(2y-1)+(4y^2+2y+1)$
$x^3-x^2(2y-1)$
$+x^2(2y-1)+6xy$
$+x^2(2y-1)-x(4y^2-4y+1)$
$+x(4y^2+2y+1)-(8y^3-1)$
$+x(4y^2+2y+1)-(8y^3-1)$
$-x+(2y+1)]$
$-x^3+6xy+(8y^3+1)\ [\ x^2+x(2y+1)+(4y^2-2y+1)$
$-x^3+x^2(2y+1)$
$-x^2(2y+1)+6xy$
$-x^2(2y+1)+x(4y^2+4y+1)$
$-x(4y^2-2y+1)+(8y^3+1)$
$-x(4y^2-2y+1)+(8y^3+1)$

Arranging in *descending* powers of y, we should have had
$-2y+(x+1)]$
$-8y^3+6xy+(x^3+1)\ [\ 4y^2+2y(x+1)+(x^2-x+1)$
$2y-(x-1)]$
$8y^3+6xy-(x^3-1)\ [\ 4y^2+2y(x-1)+(x^2+x+1).$

8. $x-(2y+3z)]$
$x^3-18xyz-(8y^3+27z^3)\ [\ x^2+x(2y+3z)+(4y^2-6yz+9z^2)$
$x^3-x^2(2y+3z)$
$+x^2(2y+3z)-18xyz$
$+x^2(2y+3z)-\ x(4y^2+12yz+9z^2)$
$+\ x(4y^2-6yz+9z^2)-(8y^3+27z^3)$
$+\ x(4y^2-6yz+9z^2)-(8y^3+27z^3)$

9. $x^2+(y^2-z^2-1)]$
$x^4+2x^2y^2+(y^4-z^4-2z^2-1)\ [\ x^2+(y^2+z^2+1)$
$x^4+x^2(y^2-z^2-1)$
$+x^2(y^2+z^2+1)+(y^4-z^4-2z^2-1)$
$+x^2(y^2+z^2+1)+(y^4-z^4-2z^2-1)$

10. $1+x)a(a-ax+ax^2+\&c.\ \ 1-3x)1+2x(1+5x+15x^2+\&c$
$a+ax$ $\qquad$ $1-3x$
$-ax$ $\qquad$ $+5x$
$-ax-ax^2$ $\qquad$ $+5x-15x^2$
$+ax^2$ $\qquad$ $+15x^2$

11. $1-2x+x^2)\ 1\ (1+2x+3x^2+\&c.$
$1-2x+x^2$
$2x-x^2$
$2x-4x^2+2x^3$
$3x^2-2x^3$

$1+bx]\ 1-ax\ [1-(a+b)x+(a+b)bx^2-\&c.$
$1+bx$
$-(a+b)x$
$-(a+b)x-(a+b)bx^2$
$(a+b)bx^2$

12. $x-a]\ x^3-px^2+qx-r\ [\ x^2+(a-p)x+(a^2-pa+q)$

$$\begin{array}{l} x^3-ax^2 \\ \hline (a-p)x^2+qx \\ (a-p)x^2-(a^2-pa)x \\ \hline (a^2-pa+q)x-r \\ (a^2-pa+q)x-(a^3-pa^2+qa) \\ \hline a^3-pa^2+qa-r \end{array}$$

Ex. 17.

1. $4x-3x=3+2$, &c.
2. $9x-3x=7+5$, &c.
3. $8x-4x=9+3$, &c.
4. $2x+5x=7-3$, &c.
5. $15x-x=-7$, &c.
6. $(m-n)x=d-a$, &c.
7. $3x-6+4=12-4x$; $\therefore\ 3x+4x=12+2$, &c.
8. $5-12+3x+12-8x=0$; $\therefore\ 5=0+5x$, &c.
9. $13x-21x+63=10-63+21x$,
   $\therefore\ 21x-13x+21x=63-10+63$, &c.
10. $5a+5x-2x=3a-15x$,
    $\therefore\ 15x+5x-2x=3a-5a$, &c.
11. $3x-9-2x+4+x-1=x+3+2x+4+3x+3$;
    $\therefore\ -2x-2x=3+3+9+1$, or $4x=-16$, &c.
12. $2x-1-6x+4+12x-9-20x+16=0$;
    $\therefore\ 2x-6x+12x-20x=1-4+9-16$, or $-12x=-10$, or $6x=5$, &c.
13. $2a+ax-6-3x=-4-2ax$;
    $\therefore\ (a-3+2a)x=-4-2a+6$, or $(3a-3)x=-(2a-2)$, or $3x=-2$, &c.
14. $m^2-mx+mn-nx=mn-mx$;
    $\therefore\ nx=m^2$, &c.

Ex. 18.

1. Mult. by 6: $3x+2x=6x-42$, &c.
2. Mult. by 12: $6x-4x=3x-12$, &c.

3. Mult. by 12: $6x-4x+3x=24-2x+5x$, &c.
4. Mult. by 15: $6x+5x-10=30x-105$, &c.
5. Mult. by 42: $12x+7x-7=42x-168$, &c.
6. Mult. by 10: $45-10x=15-7x+18$, &c.
7. Mult. by 6: $6x+28-2x=63-3x$, &c.
8. Mult. by 30: $60x-10=36-24x+15x$, &c.
9. Mult. by 154: $44x+154-126x+112=77x-847$, &c.
10. Mult. by 30: $10x-10a-12x+18b-15a+15x=0$, &c.
11. Mult. by 12: $18x-6-8x+8=3x-9-2x+10+64$, &c.
12. Mult. by 30: $5x-50=258+36x-60-10x-80$, &c.

Ex. 19.

1. Mult. by 24: $4x-6-\frac{1}{5}(72x-48)=12x-9-77$;
   $\therefore 4x-12x-6+9+77=\frac{1}{5}(72x-48)$,
   or $-8x+80=\frac{1}{5}(72x-48)$, or $\frac{1}{5}(9x-6)=-x+10$;
   mult. by 5: $9x-6=-5x+50$; &c.
2. Mult. by 45: $\frac{5}{7}(45x-405)+35x-175=81x-567+75$;
   $\therefore \frac{5}{7}(45x-405)=46x-317$;
   mult. by 7: $225x-2025=322x-2219$; &c.
3. Mult. by 24: $\frac{8}{5}(2x-1)-\frac{3}{2}(3x-2)=\frac{4}{3}(x-12)-x-12$;
   mult. by 6: $\frac{1}{5}(96x-48)-27x+18=8x-96-6x-72$;
   $\therefore \frac{1}{5}(96x-48)=29x-186$; $\therefore 96x-48=145x-930$; &c.
4. Mult. by 80: $70x+200-45x-60=24x+8-116+32x$; &c.
5. Mult. by 96: $72a-144x-128a+64x+12x-12a$
   $=45a+45x$; &c.
6. Mult. by 33: $27x-30-\frac{1}{5}(22x-77)=22x-5-x$;
   or $6x-25=\frac{1}{5}(22x-77)$; $\therefore 30x-125=22x-77$; &c.
7. Mult. by 60: $112x-28-\frac{25}{2}(2x+1)=315-\frac{15}{8}x$;
   mult. by 8: $896x-224-200x-100=2520-15x$; &c.
8. Mult. by 60: $40(a-b+x)-45(x-b+a)-48(b-a-x)$
   $=50(x+a-b)$;
   $\therefore (40-45+48-50)x=(50-40+45-48)a$
   $+(40-45+48-50)b$;
   or $-7x=7a-7b$; &c.

9. Mult. by 24: $\frac{1}{7}(96x-504)+188+56x-224$
$=24x+90-27+21x+2$;
or $\frac{1}{7}(96x-504)=-11x+101$;
$\therefore\ 96x-504=-77x+707$; &c.

10. Mult. by 80: $16x-16a-\frac{10}{3}(m-a+x)$
$=15m+15x-15a-16m$;
$\therefore\ x-a+m=\frac{10}{3}(x-a+m)$; $\therefore\ x-a+m=0$; &c.

11. Mult. by 60: $\frac{1}{5}(40x+140)-8x+28=110-9x-12$;
$\therefore\ \frac{1}{5}(40x+140)=70-x$; or $40x+140=630-9x$; &c.

12. Mult. by 144: $48x-96a+144c-70a+50x-100c$
$=24a+240c-6c+3x$;
$\therefore\ 95x=190a+190c$; &c.

Ex. 20.

1. x=the no.; $x+7=16$; $\therefore\ x=16-7$; &c.

2. x=the no. of shill. for each book; hence $10x$=the no. of shill. for 10 books, and $15x$=the no. for 15;
$\therefore\ 15x-10x=35$; &c.

3. x=the no.; $x-\frac{1}{6}x=10$; $\therefore\ 6x-x=60$; &c.

4. x=the no.; $x+10=\frac{5}{3}x$; $\therefore\ 3x+30=5x$; &c.

5. x=the greater no.; $\therefore\ x-22$=the less; then $x+(x-22)$
$=48$; &c.

6. x=the no. of years for the older, $\therefore\ 49-x$ for the younger; then $x-(49-x)=13$, or $2x=49+13$; &c.

7. x=the no. of votes on the successful side; $\therefore\ x-47$=the no. of other votes; then $x+(x-47)=979$; &c.

8. x=the no.; then $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x-\frac{1}{5}x=106$;
mult. by 60: $(30+20+15-12)x=106\times60$, or $53x$
$=106\times60$; &c.

9. x=no. of shill. in total amounts; $x-\frac{1}{2}x-\frac{1}{3}x-\frac{1}{12}x=21\times15$;
mult. by 12: $(12-6-4-1)x=21\times15\times12$,
or $x=21\times15\times12$ shill.$=21\times3\times3$ pounds.

10. x=whole no. of feet long; $\frac{1}{4}x+\frac{1}{3}x+10=x$, or $3x+4x+120=12x$; &c.

11. x=whole no. of eggs; $2x+\frac{1}{2}x+1\frac{1}{2}=104$, or $4x+x+3=208$; &c.

12. $2x$=one, $3x$=the other; $2x+3x=150$; &c.
Or thus: x=one no.; $150-x$=the other;
$x=\frac{2}{3}(150-x)$, or $3x=300-2x$; &c.

13. $2x$=the no.; $4x+8=5x$; &c.

14. x, $x-7$, $x-17$, the nos.; sum, $3x-24=87$; &c.

15. x pounds to the 1st, $\frac{1}{3}x$ to the 2nd, $\frac{1}{3}(x+\frac{1}{3}x)$ to the 3rd; $x+\frac{1}{3}x+\frac{1}{3}x+\frac{1}{9}x=64$; $\therefore (9+3+3+1)x=64\times 9$, or $16x=64\times 9$; &c.

16. x pounds for the 1st, $x+12$ for the 2nd, $x+18$ for the 3rd, $x+20$ for the 4th; sum, $4x+50=230$; &c.

17. x yds. at $11\frac{1}{2}s.$ amounts to $11\frac{1}{2}x$ shill.; $20-x$ yds. at $7\frac{1}{2}s.$ amounts to $150-7\frac{1}{2}x$ shill.; then $11\frac{1}{2}x+150-7\frac{1}{2}x=21\times 10$; $\therefore 23x-15x=420-300$; &c.

18. x yds. at $12s.$ exceeds $(449-x)$ yds. at $17s.$ by $23s.$; $\therefore 12x-17(449-x)=23$, or $12x-7633+17x=23$; &c.

19. If they meet in x hours, one has travelled $9\frac{1}{2}x$ miles, the other $9\frac{1}{4}x$ miles; hence $9\frac{1}{2}x+9\frac{1}{4}x=200$; $\therefore 38x+37x=800$; whence $x=10\frac{8}{3}$ hrs.; &c.

20. A's present age$=2x$ yrs.; B's$=x$ yrs.; and 22 yrs. ago A's age $=2x-22$, and B's $=x-22$; $\therefore 2x-22=3(x-22)$; &c.

21. x years hence their ages will be $40+x$ and $8+x$; and if $40+x=3(8+x)$, we have $x=8$.

22. x=the no.; 1st remr. $2x-10$; 2nd remr. $2(2x-10)-20=40$; hence $(x-5)-5=10$, or $x=20$.

23. x=no. of pence given for each orange; 3 oranges cost $3x$ pence, and 9 oranges cost $9x$ pence; $\therefore 12-3x$

$= 9x - 12$; whence $x = 2$ pence, and $30d. \div 2d. = 15$ oranges.

24. x shill. each at first; when A won he had $x+30$, and B had $x-30$; $\therefore\ 7(x+30) = 13(x-30)$, or $13x - 390 = 7x + 210$; &c.

25. If A has x shill. at first, then if he win the stake he will have $x+5$, and B will have $\frac{1}{3}(x+5)$; hence B at first has $\frac{1}{3}(x+5)+5$. But if A lose, he will then have $x-5$, and B will have $\frac{1}{3}(x+5)+10$; $\therefore\ x-5 = \frac{2}{3}(x+5)+20$; hence $3x-15 = 2x+10+60$; &c.

26. x days idle, $28-x$ days at work; he receives for $28-x$ days at $2\frac{1}{2}s.$, viz. $2\frac{1}{2}(28-x)$ shill., and pays for x days at a shilling, viz. x shill.; so that he receives altogether $2\frac{1}{2}(28-x)-x = 52\frac{1}{2}$; hence $140-5x-2x = 105$; &c.

27. x gall. through 1st, $x-15$ through 2nd, $x-10$ through 3rd, per minute; $20(x+x-15+x-10) = 820$, or $3x-25 = 41$; &c.

28. If B has walked for x hrs., and therefore A for $x+\frac{1}{4}$ hrs., we have the whole no. of miles $4\frac{3}{4}x = 4(x+\frac{1}{4})$, or $19x = 16x+4$; &c.

29. x = no. of men added, when 20×1000 rations were in store; hence $5(1000+x) = 20000$; $\therefore\ 1000+x = 4000$; &c.

30. x lb. at $2\frac{1}{2}s.$ + 30 lb. at $3\frac{1}{3}s.$ = $(x+30)$ lb. at $3s.$; $\therefore\ 2\frac{1}{2}x + 100 = 3x + 90$, or $\frac{1}{2}x = 10$; &c.

31. If A's no. of shill. $= x$, then B's $= 8-x$, and C's $= 10-x$; $\therefore$ B and C together have $8-x+10-x = 12$; &c.

32. $8x$ = his original money, in shillings; he lost $3x + 1\frac{1}{2}$, and had remaining $5x-1\frac{1}{2}$; then by winning he had $1\frac{3}{7}(5x-1\frac{1}{2})-1\frac{1}{2}$, and then by lossing he had

$\frac{2}{3}$ of $\{1\frac{3}{7}(5x-1\frac{1}{2})-1\frac{1}{2}\} = 4x+1$,

or $\frac{2}{3}$ of $\frac{1}{7}(50x-15)-1 = 4x+1$,

or $\frac{1}{21}(50x-15) = 2x+1$; or $50x-15 = 42x+21$; &c.

Ex. 27.

1.

$$\begin{array}{r|l} 3x^2 & x^3+6x^2y+12xy^2+8y^3(\\ 2y(3x+2y)=6xy+4y^2 & x^3 \qquad\qquad x+2y.\ \textit{Ans.} \\ \hline 3x^2+6xy+4y^2 & \quad 6x^2y+12xy^2+8y^3 \\ & \quad 6x^2y+12xy^2+8y^3 \\ & \quad \hline \end{array}$$

2.

$$\begin{array}{r|l} 3a^2 & a^3-9a^2+27a-27(\\ -3(3a-3)=-9a+9 & a^3 \qquad\qquad a-3.\ \textit{Ans.} \\ \hline 3a^2-9a+9 & \quad -9a^2+27a-27 \\ & \quad -9a^2+27a-27 \\ & \quad \hline \end{array}$$

3.

$$\begin{array}{r|l} 3x^2 & x^3+12x^2+48x+64(\\ 4(3x+4)=12x+16 & x^3 \qquad\qquad x+4.\ \textit{Ans.} \\ \hline 3x^2+12x+16 & \quad 12x^2+48x+64 \\ & \quad 12x^2+48x+64 \\ & \quad \hline \end{array}$$

4.

$$\begin{array}{r|l} 12a^2 & 8a^3-36a^2b+54ab^2-27b^3(\\ -3b(6a-3b)=-18ab+9b^2 & 8a^3 \qquad\qquad 2a-3b.\ \textit{Ans.} \\ \hline 12a^2-18ab+9b^2 & \quad -36a^2b+54ab^2-27b^3 \\ & \quad -36a^2b+54ab^2-27b^3 \\ & \quad \hline \end{array}$$

Ex. 28.

1.

$$
\begin{array}{r}
3a^4 \\
2a(3a^2+2a)=6a^3+4a^2 \\
\hline
3a^4+6a^3+4a^2 \\
(2a)^2= \qquad 4a^2 \\
\hline
3a^4+12a^3+12a^2 \\
1(3a^2+6a+1)= \qquad 3a^2+6a+1 \\
\hline
3a^4+12a^3+15a^2+6a+1
\end{array}
\qquad
\begin{array}{l|l}
a^6+6a^5+15a^4+20a^3+15a^2+6a+1(& \\
a^6 & a^3+2a+1. \ \textit{Ans.} \\
\hline
6a^5+15a^4+20a^3 & \\
6a^5+12a^4+\ 8a^3 & \\
\hline
3a^4+12a^3+15a^2+6a+1 & \\
3a^4+12a^3+15a^2+6a+1 & \\
\hline
\end{array}
$$

2.

$$
\begin{array}{r}
3x^4 \\
-4x(3x^2-4x)=-12x^3+16x^2 \\
\hline
3x^4-12x^3+16x^2 \\
(-4x)^2= \qquad 16x^2 \\
\hline
3x^4-24x^3+48x^2 \\
2(3x^2-12x+2)= \qquad 6x^2-24x+4 \\
\hline
3x^4-24x^3+54x^2-24x+4
\end{array}
\qquad
\begin{array}{l|l}
x^6-12x^5+54x^4-112x^3+108x^2-48x+8(& \\
x^6 & x^3-4x+2. \ \textit{Ans.} \\
\hline
-12x^5+54x^4-112x^3 & \\
-12x^5+48x^4-\ 64x^3 & \\
\hline
6x^4-\ 48x^3+108x^2-48x+8 & \\
6x^4+\ 48x^3+108x^3-48x+8 & \\
\hline
\end{array}
$$

6. In this example the work of constructing the partial and complete divisors is subjoined to the other work.

$$\begin{array}{l|l} x^9-3x^8+6x^7-10x^6+12x^5-12x^4+10x^3-6x^2+3x-1 & \\ x^9 & x^3-x^2+x-1. \ \textit{Ans.} \\ \hline -3x^8+6x^7-10x^6 & \\ -3x^8+3x^7-\ x^6 & \\ \hline 3x^7-9x^6+12x^5-12x^4+10x^3 & \\ 3x^7-6x^6-\ 6x^5-\ 3x^4+\ x^3 & \\ \hline -3x^6+\ 6x^5-\ 9x^4+\ 9x^3-6x^2+3x-1 & \\ -3x^6+\ 6x^5-\ 9x^4+\ 9x^3-6x^2+3x-1 & \\ \hline \end{array}$$

$$\left.\begin{array}{rl} & 3x^6 \\ -x^2(3x^3-x^2)= & -3x^6+\ x^4 \\ \hline & 3x^6-3x^5+\ x^4 \\ (-x^2)^2= & \ x^4 \\ \hline & 3x^6-6x^5+3x^4 \end{array}\right\}$$

$$\left.\begin{array}{rl} x(3x^3-3x^2+x)= & 3x^4-3x^3+\ x^2 \\ \hline & 3x^6-6x^5+6x^4-3x^3+\ x^2 \\ (+x)^2= & \ x^2 \\ \hline & 3x^6-6x^5+9x^4-6x^3+3x^2 \end{array}\right\}$$

$$\begin{array}{rl} -(3x^3-3x^2+3x-1)= & -3x^3+3x^2-3x+1 \\ \hline & 3x^6-6x^5+9x^4-9x^3+6x^2-3x+1 \end{array}$$

Ex. 33.

1. First quotient $a-2x$; remr. $3ax^2+3x^3$, from which strike out $3x^2$, giving $a+x$ as divisor for $a^2+2ax+x^2$.
2. First quotient 1; remr. $-4x+4$, from which strike out -4, giving $x-1$ as divisor for x^2+x-2.
3. Here we have $2(x^3+3x^2+3x+1)$ and $6(x^3+x^2-x-1)$; and 2 and 6 have a common factor 2, which must be multiplied into the G.C.M. of the bracketed quantities, giving $2(x^2+2x+1)$.

4. Throwing out the factors $2y$ and 3, as they have no C.M., we have to find the G.C.M. of y^2-5y+6 and $y^4-5y^3+8y^2-8$;

$y^2-5y+6)\ y^4-5y^3+8y^2-8\ (y^2+2$;

strike out $10)\ 10y-20$, remainder;

$y-2$ divides y^2-5y+6.

5. Cancel the factor x^2 from the 2nd quantity:

$x^2-4a^2)\ x^3-6ax^2+12a^2x-8a^3\ (x-6a$;

strike out $16a^2)\ 16a^2x-32a^3$, remainder;

$x-2a$ divides x^2-4a^2.

6. Cancel the factor 2 from the 1st quantity:

$x^3+x^2+7x+39)\ x^3+5x^2+7x+3\ (1$;

strike out $4)\ 4x^2-36$, remainder;

$x^2-9)\ x^3+x^2+7x+39\ (x+1$;

strike out $16)\ 16x+48$, remainder;

$x+3$ divides x^2-9.

7. In $3(x^3+x^2-5x+3)$ and $3x(x^3+x^2-7x-3)$ the factors 3 and $3x$ contain the C.M. 3, to be reserved;

$x^3+x^2-5x+3)\ x^3+x^2-7x-3\ (1$;

strike out $-2)\ -2x-6$, remainder;

$x+3$ divides x^3+x^2-5x+3;

$\therefore$ G.C.M. $=3(x+3)$.

8. $x^3+x^2y+xy^2+y^3)\ x^4+x^3y+xy^3-y^4\ (x$;

strike out $-y^2)\ -x^2y^2-y^4$, remainder;

x^2+y^2 measures the 1st divisor.

9. Each quantity gives out a factor a, to be reserved;

$2a^3+a^2b-4ab^2-3b^3)\ 4a^3+a^2b-2ab^2+b^3\ (2$;

strike out $-b)\ -a^2b+6ab^2+7b^3$

$a^2-6ab-7b^2$

$a^2-6ab-7b^2)\ 2a^3+a^2b-4ab^2-3b^3\ (2a+13b$;

strike out $88b^2)\ 88ab^2+88b^3$

$a+b$ divides $a^2-6ab-7b^2$;

$\therefore$ G.C.M. $=a(a+b)$.

10. Here the quantities give out factors $3a^2$ and $10a$, which have the C.M. a, to be reserved.

$a^4+5a^3b-ab^2-5b^3)\ a^4-3a^3b-ab^2+3b^3\ (1$;

strike out $-8b)\ 8a^3b+8b^3$, remainder;

a^3-b^2 divides the 1st divisor:

$\therefore$ G.C.M. $=a(a^3-b^3)$.

11. $x^4-2x^3y+2xy^3-y^4)\ x^4-2x^3y+2x^2y^2-2xy^3+y^4\ (1$;

striking out $2y^2$ from the remainder, $2x^2y^2-4xy^3+2y^4$ gives $x^2-2xy+y^2$, which divides the 1st divisor.

12. Making x^4+6x^3+ &c. the divisor, and striking out $-4x$ from the remainder, we obtain x^2+4x+4, which measures the divisor.

Ex. 34.

1.

$$8x^2+6x-9$$
$$3$$

$6x^2+13x+6)\ 24x^2+18x-27\ (4$;

strike out $-17)\ -34x-51$, remainder;

$2x+3$ divides the 1st divisor.

2. $15x^2-x-6$

3

$9x^2-3x-2)\ 45x^2-3x-18\ (5$;

strike out 4) $12x-8$, remainder;

$3x-2$ divides the 1st divisor.

3. Cancel the factor x from the 2nd quantity:

$21x^2-26x+8$

2

$6x^2-x-2)\ 42x^2-52x+16\ (7$;

strike out $-15)\ -45x+30$, remainder;

$3x-2$ divides the 1st divisor.

4. The quantities give out factors 2 and 3, which have no C.M.

$3x^3-3x^2+x-1$

4

$4x^2-5x+1)\ 12x^3-12x^2+4x-4\ (3x$;

$3x^2+x-4$, remainder;

4

$12x^2+4x-16\ (3$;

strike out 19) $19x-19$, remainder;

$x-1$ divides the 1st divisor.

5. Cancel the factor x from the 2nd quantity:

$5x^3+\ x^2-\ 54x+18$

3

$3x^3-22x-15)\ 15x^3+3x^2-162x+54\ (5$;

remr. $3x^2-52x+129)\ 3x^3-\ 22x-15\ (x$;

$52x^2-151x-15$, remainder;

3

$156x^2-453x-45\ (52$;

strike out 2251) $2251x-6753$, remainder;

$x-3$ divides the 2nd divisor.

6. Strike out the factor x from the 2nd quantity:

$$3x^3 - 3x^2y + xy^2 - y^3$$
$$4$$
$$4x^2 - xy - 3y^2) \; 12x^3 - 12x^2y + 4xy^2 - 4y^3 \; (3x;$$
$$- 9x^2y + 13xy^2 - 4y^3, \text{ remainder;}$$
$$4$$
$$-36x^2y + 52xy^2 - 16y^3 \; (-9y;$$

strike out $43y^2$) $43xy^2 - 43y^3$, remainder;

$x - y$ divides the 1st divisor.

7. $x^3 - 8x + 3) \; x^6 + 3x^5 + x + 3 \; (x^3 + 3x^2 + 8x + 21;$

strike out 5) $55x^2 + 145x - 60$, remainder;

$11x^2 + 29x - 12$

$$x^3 - 8x + 3$$
$$11$$
$$11x^2 + 29x - 12) \; 11x^3 - 88x + 33 \; (x;$$
$$- 29x^2 - 76x + 33$$
$$11$$
$$-319x^2 - 836x + 363 \; (-29;$$

strike out 5) $5x + 15$, remainder;

$x + 3$ measures the 2nd divisor.

8.
$$- 7x^3 + 4x^2 + 21x - 12$$
$$5$$
$$5x^3 + 2x^2 - 15x - 6) \; -35x^3 + 20x^2 + 105x - 60 \; (-7;$$

strike out 34) $34x^2 - 102$, remainder;

$x^2 - 3$ divides the 1st divisor.

9. $20x^4+x^2-1)\ 25x^4+5x^3-x-1\ (1$;

strike out x) $5x^4+5x^3-x^2-x$, remainder;

$5x^3+5x^2-x-1)\ 20x^4+x^2-1\ (4x-4$;

strike out 5) $25x^2-5$, remainder;

$5x^2-1$ divides the 2nd divisor.

10. The 2nd quantity divided by the 1st leaves $3x^4-2x^3y+x^2y^2$, from which strike out x^2, and we have $3x^3-2xy+y^2$, which divides the 1st quantity.

11. The quantities give out facters $3x$ and $2x$, which have the C.M. x, to be reserved.

$6x^4+4x^3y-\ 9x^2y^2-3xy^3+2y^4$

2

$4x^4-4x^2y^2+4xy^3-y^4)\ 12x^4+8x^3y-18x^2y^2-\ 6xy^3+4y^4\ (3$;

strike out y) $8x^3y-\ 6x^2y^2-18xy^3+7y^4$, remr.;

$8x^3-6x^2y-18xy^2+7y^3)\ 8x^4-8x^2y^2+8xy^3-2y^4\ (x$;

remr. $\times 4=24x^3y+40x^2y^2+4xy^3-8y^4\ (3y$;

strike out $29y^2$) $58x^2y^2+58xy^3-29y^4$, remainder;

$2x^2+2xy-y^2$ divides the 2nd divisor;

$\therefore$ G.C.M. $=x(2x^2+2xy-y^2)$.

12. $4x^3-12x^2+9x-1)\ 4x^4-8x^3+4x^2-32x+32\ (x+1$,

$7x^2-40x+33$, remainder;

$7x^2-40x+33)\ 28x^3-84x^2+\ 63x-\ 7\ (4x$;

remr. $\times 7=532x^2-483x-49\ (76$;

strike out 2557) $2557x-2557$, remainder;

$x-1$ divides the 2nd divisor.

Ex. 35.

1. $2abc$ into $2a$ and $3b = 2abc \times 2a \times 3b =$ &c.
   $3xy$ into $3x^2$ and $4y^2 = 3xy \times 3x^2 \times 4y^2 =$ &c.
   ay into x and $x-y = ay \times x(x-y) =$ &c.
   a into $b+d$ and $b-d =$ &c.

2. L.C.M. of the nos. is 120, and of a^4, a^3b, a^2b^2, is a^4b^2; &c.
   L.C.M. of the nos. is 10, and of a^5, a^4b, a^3b^2, &c. is a^5b^5; &c.
   L.C.M. of the nos. is 1800, and x^2, ax, a^2, &c. is a^3x^3; &c.

3. $(a+b)$ into 2 and $3(a-b) = 6(a+b)(a-b) =$ &c.
   L.C.M. of the nos. is 12; $12a(a-1)(a+1) =$ &c.
   L.C.M. of the nos. is 120, and of $x(x+y)$, $y(x-y)$, and (x^2-y^2) is $xy(x^2-y^2)$.

4. L.C.M. of $4a(a^2-b^2)$, $12b^2(a+b)$, and $8a^2(a-b)$ is $24a^2b^2(a^2-b^2)$.
   L.C.M. of $6xy(x+y)$, $9x(x^2-y^2)$, and $4y^2(x+y)$ is $36xy^2(x^2-y^2)$.

Ex. 36.

1. $\frac{xy(a+y)}{axy} =$ &c. $\frac{x(c+x)}{a^2(c+x)} =$ &c. $\frac{11m(m+2x)}{33(m+2x)(m-2x)} =$ &c.
   $\frac{7x(2x-y)}{5a(2x-y)} =$ &c. $\frac{5a^2b(a-3b)}{10ab^2(2b+a)} =$ &c.

2. $\frac{6x(x^2-3y^2)}{6xy(x-2y)} =$ &c. $\frac{4m^2n^2}{2mn(m+n)} =$ &c. $\frac{3a^2b^2c^2}{abc(a+b+c)} =$ &c.
   $\frac{3xy^2(3x-5y)}{3xy^2(4x-7y)} =$ &c.

3. $\frac{c(ab+9b-5c)}{2df(ab+9b-5c)} =$ &c. $\frac{c(a+b)+y(a+b)}{f(a+b)+2x(a+b)} =$ &c.
   $\frac{acx^2+adx-bcx-bd}{(ax+b)(ax-b)} = \frac{cx(ax-b)+d(ax-b)}{(ax+b)(ax-b)} =$ &c.

4. $\frac{(x+1)(x-1)}{a(x+1)}$=&c. $\frac{(x^2+a^2)(x^2-a^2)}{x^3(a^2-x^2)}$=&c.

$\frac{1}{a^2+b^2}\times\frac{a^6-b^6}{a^2-b^2}=\frac{a^4+a^2b^2+b^4}{a^2+b^2}$. $\frac{x(x^2-b^2)}{(x+b)(x+b)}$=&c.

$\frac{a(a-b)+x(a-b)}{a(a+x)+b(a+x)}=\frac{(a+x)(a-b)}{(a+x)(a+b)}$=&c.

5. $\frac{(x-1)(x-3)}{(x+1)(x-3)}$=&c. 6. $\frac{(x+3)(x-1)}{(x+3)(x+2)}$=&c.

7. $\frac{(a-2b)(a+b)}{(a-2b)(a-b)}$=&c. 8. $\frac{(2a-3x)(3a-2x)}{(2a-3x)(5a+3x)}$=&c.

9. $\frac{(3a+x)(2a-3x)}{(3a+x)(2a+3x)}$=&c.

10. Divide numerator and denominator by their G.C.M., which is readily found to be $x-3$.

Or proceed thus: $\frac{(x^2-3x)+(4x-12)}{x(x^2-5x+6)+(x-3)}$

$=\frac{x(x-3)+4(x-3)}{x(x-2)(x-3)+(x-3)}=\frac{(x+4)(x-3)}{(x^2-2x+1)(x-3)}$=&c.

11. The G.C.M. may be found$=x-3y$. Or we may proceed thus:

$$\frac{(7x^2-21xy)-(2xy-6y^2)}{(5x^2-15x^2y)-(3x^2y-9xy^2)+(2xy^2-6y^3)}$$

$=\frac{(7x-2y)(x-3y)}{(5x^2-3xy+2y^2)(x-3y)}$=&c.

12. $\frac{5a^3(a^2+2ax+x^2)}{x(a^3+2a^2x+2ax^2+x^3)}=\frac{5a^3(a+x)^2}{x(a+x)(a^2+ax+x^2)}$=&c.

13. $\frac{(x^3-x^2)+(4x^2-4)}{(x-1)(x^2+x+1)}=\frac{x^2(x-1)+4(x+1)(x-1)}{(x-1)(x^2+x+1)}$

$=\frac{(x^2+4x+4)(x-1)}{(x-1)(x^2+x+1)}$=&c.

14. $\frac{(x^3-1)-(3x-3)}{(x^3-1)+(4x^2-4)}=\frac{(x^2+x+1)-3}{(x^2+x+1)+4(x+1)}$=&c.

15. Divide numerator and denominator by their G.C.M., which is x^2+ax+a^2, as here obtained:

$x^4+ax^3-a^3x-a^4)\ x^4+a^2x^2+a^4\ (1$;

strike out $-a)\ -ax^3+a^2x^2+a^3x+2a^4$, remainder;

$x^3-ax^2-a^2x-2a^3)\ x^4+ax^3-a^3x-a^4\ (x+2a$;

strike out $3a^2)\ 3a^2x^2+3a^3x+3a^4$, remainder;

x^2+ax+a^2 measures the 2nd divisor.

16. $$\frac{(2a^2x^4-2ax^3)+(a^2x^4-1)}{(4a^3x^6-3a^2x^4-ax^2)+(a^2x^4-2ax^2+1)}$$
$$=\frac{2ax^2(ax^2-1)+(ax^2+1)(ax^2-1)}{(4a^2x^4+ax^2)(ax^2-1)+(ax^2-1)^2}$$
$$=\frac{(3ax^2+1)(ax^2-1)}{(4a^2x^4+2ax^2-1)(ax^2-1)}=\&c.$$

Ex. 37.

3. $$\frac{a^2+2ab+b^2-c^2}{2ab}=\frac{(a+b)^2-c^2}{2ab}=\frac{\{(a+b)+c\}\{(a+b)-c\}}{2ab}.$$
$$\frac{2ab-a^2-b^2+c^2}{2ab}=\frac{c^2-(a-b)^2}{2ab}=\frac{\{c+(a-b)\}\{c-(a-b)\}}{2ab}.$$

4. $$\left(a+\frac{a^2+b^2-c^2}{2b}\right)\times\left(a-\frac{a^2+b^2-c^2}{2b}\right)$$
$$=\frac{(a+b)^2-c^2}{2b}\times\frac{c^2-(a-b)^2}{2b};\ \&c., \text{ as above.}$$

Ex. 41.

1. $\frac{2}{5}$ of (i)$=\frac{4-3x}{10}$; $\frac{3}{5}$ of (ii)$=\frac{3x}{15-2x}$; $\frac{2}{7}$ of (iii)$=\frac{6-2x}{2x+5}$;

$\frac{6}{7}$ of (iv)$=\frac{18x+14}{21}$; $\frac{13}{11}$ of (v)$=\frac{27-4x}{8x-18}$;

$\frac{12}{11}$ of (vi)$=\frac{12x-40}{33-2x}$.

2. $\frac{2}{3}$ of (i)$=\dfrac{2x-5(3x-2)}{6}=$&c.

$\frac{4}{4}$ of (ii)$=\dfrac{24x-3(3+5x)}{9}=$&c.

$\frac{6}{6}$ of (iii)$=\dfrac{14-3(x-2)}{2(x+1)-27}=$&c.

$\frac{30}{30}$ of (iv)$=\dfrac{54-20(x+2)}{9(x+1)}=$&c.

3. (i) $\times\dfrac{1+x}{1+x}=\dfrac{1}{(1+x)-1}=$&c.

(ii) $\times\dfrac{1-x^2}{1-x^2}=\dfrac{(1-x)+x(1+x)}{(1+x)-x(1-x)}=$&c.

(iii) $\dfrac{b^2\left(\dfrac{a^2+b^2}{2a^2b^2}-\dfrac{2}{a^2+b^2}\right)}{a^2\left(\dfrac{a^2+b^2}{2a^2b^2}-\dfrac{2}{a^2+b^2}\right)}=$&c.

(iv) $\times\dfrac{a^2-x^2}{a^2-x^2}=\dfrac{(a+x)^2+(a-x)^2}{(a+x)^2-(a-x)^2}=\dfrac{2(a^2+x^2)}{4ax}=$&c.

4. (i) $\dfrac{x(1+xy)-x+y}{1+xy+x(x-y)}=\dfrac{y(x^2+1)}{x^2+1}=$&c.

(ii) $\dfrac{1}{x-1+\dfrac{4-x}{4-x+x}}=\dfrac{4}{4x-4+4-x}=$&c.

(iii) $\dfrac{x}{1-\dfrac{x(1-x+x^2)}{1+x^3+x}}=\dfrac{x(1+x+x^3)}{1+x^3+x-x+x^2-x^3}=$&c.

Ex. 42

1. $\dfrac{x(1-x^4)}{1-x^2}=x(1+x^2)=1\times(1+1)=2.$

2. $\dfrac{(x+7)(x-5)}{(x-1)(x-5)}=\dfrac{x+7}{x-1}=\dfrac{5+7}{5-1}=3.$

3. $\frac{(x+3)(x-2)}{(x+3)(x-1)}=\frac{x-2}{x-1}=\frac{-3-2}{-3-1}=\frac{5}{4}$.

4. $\frac{x^2(x+2)-4(x+2)}{x^2(x+2)+(x^2-4)}=\frac{x^2-4}{x^2+x-2}=\frac{(x+2)(x-2)}{(x-1)(x+2)}=\frac{x+2}{x-1}=\frac{4}{3}$.

5. $\frac{(5x-1)(x-1)}{(5x-1)(-x-1)}=\frac{1-x}{1+x}=\frac{1-\frac{1}{5}}{1+\frac{1}{5}}=\frac{4}{6}=$ &c.

6. $\frac{x^3-a^3}{(x^3-a^3)-2ax(x-a)}=\frac{a^2+ax+a^2}{(x^2+ax+a^2)-2ax}=\frac{3a^2}{3a^2-2a^2}$
$=$ &c.

7. $\frac{(2x^2-3x-6)(x-2)}{(x^2+2x-3)(x-2)}=\frac{2x^2-3x-6}{x^2+2x-3}=\frac{8-6-6}{4+4-3}=$ &c.

8. $\frac{(x^2+3x-10)(x-3)}{(x^2-2x-6)(x-3)}=\frac{x^2+3x-10}{x^2-2x-6}=\frac{9+9-10}{9-6-6}=$ &c.

9. $\frac{(2x+a)(x-2a)}{(2x+a)(x^2+2ax-a^2)}=\frac{x-2a}{x^2+2ax-a^2}=\frac{-\frac{1}{2}a-2a}{\frac{1}{4}a^2-2a^2}=$ &c.

10. $\frac{(3x^3+x^2)+(3x^2+x)+(9x+3)}{(3x^3+x^2)-(3x^2+x)+(9x+3)}=\frac{x^2+x+3}{x^2-x+3}=$ &c.

11. $\frac{2-(x+1)}{x^2-1}=-\frac{x-1}{(x-1)(x+1)}=-\frac{1}{x+1}=-\frac{1}{2}$.

12. $\frac{a(x^2-2cx+c^2)}{b(x^2-2cx+c^2)}=\frac{a}{b}$.

Ex. 43.

1. Mult. by ac; then $a^2c-bcx=a^2x-ab^2$;
hence $(a^2+bc)x=a^2c+ab^2$.

2. Mult. by cex: then $ace+bex-cdx=0$;
hence $(cd-be)x=ace$.

3. Mult. by ab: then $bx+ax=abc$;
hence $(a+b)x=abc$.

4. Mult. by bce: then $abex+b^2e-ace=bc^2x+bcd$;
hence $(abe-bc^2)x=bcd-b^2e+ace$.

5. Mult. by dx: then $ad^2+ax^2=acdx+ax^2$;
hence $acdx=ad^2$, or $cx=d$.

6. Mult. by abx: then $a^2-b^2=abx(a^2-b^2)$;
hence $abx=1$.

7. Mult. by bf: then $afx+bcx=bfgx+bfh-bcx$;
hence $(af+2bc-bfg)x=bfh$.

8. Mult. by 12: then $16a(1+x)-9(a-x)$
$=9a(1-x)-16(a+x)$;
or $16a+16ax-9a+9x=9a-9ax-16a-16x$;
hence $(25a+25)x=-14a$.

9. Mult. by $12x$: then $6x^2+9x+16x^2$
$=12+24x^2+8x-2x^2-2x$;
hence $3x=12$.

10. Mult. by 40: then $40-5x\left(4-\frac{3}{x}\right)=\frac{40}{3}(6-5x)+213$;

hence $40-20x+15=80-\frac{200x}{3}+213$;

or $-238-20x=-\frac{200x}{3}$; $\therefore\ 714+60x=200x$.

Ex. 44.

1. Mult. by 10: then $\frac{10x}{7}+\frac{8}{7}-5x=\frac{8}{7x}-\frac{25x}{7}$;

hence $\frac{35x}{7}+\frac{8}{7}-5x=\frac{8}{7x}$; or $\frac{8}{7}=\frac{8}{7x}$; $\therefore\ \frac{1}{x}=1$.

2. $\frac{4x}{6}-\frac{2-x}{8x}=\frac{3x}{6}-\frac{1}{2}+\frac{x}{6}$; or $\frac{2-x}{8x}=\frac{1}{2}$;

hence $8x=4-2x$; &c.

3. $\frac{8x}{14}+\frac{5}{14}+\frac{7x-3}{6x+2}=\frac{8x}{14}+\frac{12}{14}$; or $\frac{7x-3}{6x+2}=\frac{1}{2}$;

hence $7x-3=3x+1$.

4. Mult. by $x+3$: then $8x+6+\frac{3x+9}{x+1}=8x+24$;

hence $\frac{x+3}{x+1}=6$; &c.

5. Mult. by $x+c$: then $\left(\frac{a}{b}+\frac{b}{a}\right)x=x+c$;

hence $\left(\frac{a^2+b^2}{ab}-1\right)x=c$; and $x=\frac{abc}{a^2-ab+b^2}$.

6. Mult. by 2: then $\frac{2x-6}{x+2}=1+\frac{2x-6}{2x-1}$;

that is, $2-\frac{10}{x+2}=1+1-\frac{5}{2x-1}$; hence $\frac{2}{x+2}=\frac{1}{2x-1}$.

7. $12x^2-4ax-a^2=12x^2-bx-b^2$;

hence $(b-4a)x=a^2-b^2$.

8. $\frac{3x-x+1}{9}+\frac{31}{36}=\frac{12-x+2}{20}$; or $\frac{2x}{9}+\frac{35}{36}=\frac{7}{10}-\frac{x}{20}$;

hence $\frac{40x+9x}{180}=\frac{126-175}{180}$; $\therefore 49x=-49$.

9. Mult. by 6: then $\frac{8x-6}{x-3}+\frac{3}{1-x}=8$;

or $8+\frac{18}{x-3}+\frac{3}{1-x}=8$; $\therefore \frac{1}{1-x}=\frac{-6}{x-3}$;

hence $x-3=-6+6x$.

10. Mult. by ab: then $\frac{b}{b-x}+\frac{a}{c-x}=\frac{b}{c-x}$;

$\therefore \frac{b}{b-x}=\frac{b-a}{c-x}$; hence $bc=b^2-ab+ax$;

$\therefore ax=b(a-b+c)$.

11. $\frac{2x+3}{2x+1}+\frac{1}{3x^2}=1+\frac{1}{x}$; or $1+\frac{2}{2x+1}=1+\frac{3x-1}{3x^2}$

$\therefore 6x^2=6x^2+x-1$; whence $x=1$.

12. Mult. by 6: then $\frac{4x+2a}{x-a}+\frac{9x-3a}{x+a}=13$;

or $4+\frac{6a}{x-a}+9-\frac{12a}{x+a}=13$;

hence $\frac{1}{x-a}=\frac{2}{x+a}$; $\therefore\ 2x-2a=x+a$.

13. $\frac{3x}{4}-\frac{1}{6}-\frac{x}{6}+\frac{10-2x}{55}=\frac{60+x-1}{55}$;

$\frac{9x-2-2x}{12}=\frac{49+3x}{55}$; $\therefore\ 55(7x-2)=12(49+3x)$.

14. Mult. by $2x$: then $\frac{2x^2}{a+x}=2a+2x-2a+b$;

$\therefore\ 2x^2=(a+x)(b+2x)$; or $2x^2=ab+(2a+b)x+2x^2$;

hence $(2a+b)x=-ab$.

15. Mult. by 6: then $\frac{6x+24}{3x+5}+7=\frac{18x+48}{2x+3}$;

or $2+\frac{14}{3x+5}+7=9+\frac{21}{2x+3}$; whence $\frac{2}{3x+5}=\frac{3}{2x+3}$;

$\therefore\ 9x+15=4x+6$.

16. $\frac{1}{25}(11x-13)+\frac{1}{7}(19x+3)-\frac{1}{12}(15x-76)=\frac{587-17x}{21}$;

mult. by 84:

$\frac{84}{25}(11x-13)+228x+36-105x+532=2348-68x$;

$\therefore\ \frac{84}{25}(11x-13)+191x=1780$;

mult. by 25: then $924x-1092+4775x=44500$;

whence $5699x=45592$.

17. Mult. by 18: then $10x+17-\frac{216x+36}{13x-16}=10x-8$;

$\therefore\ 25(13x-16)=216x+36$.

18. $\frac{2x+3}{6}-\frac{30-3x}{11}=\frac{12-2x-3}{33}$;

$\therefore \frac{2x+3}{6}=\frac{90-9x+9-2x}{33}=3-\frac{x}{3}$;

hence $2x+3=18-2x$.

19. Mult. by 52: then

$26x-51-8+24x=52x-\frac{1}{3}(20x-10+3x)$;

$\therefore 2x+59=\frac{1}{3}(23x-10)$; or $6x+177=23x-10$.

20. Mult. by 15: then $6x+13-\frac{9x+15}{x-5}=6x$;

$\therefore 13(x-5)=9x+15$.

21. $\frac{2x-14}{x+7}=\frac{2x-15}{x-3}-\frac{1}{x+7}$, or $\frac{2x-13}{x+7}=\frac{2x-15}{x-3}$;

or $2-\frac{27}{x+7}=2-\frac{9}{x-3}$; $\therefore 3(x-3)=x+7$.

22. $44-\frac{43}{3x+1}+8+\frac{13}{x-1}=52$;

$\therefore 43(x-1)=13(3x+1)$.

23. Mult. by 9: then $\frac{63x+9}{x-1}=\frac{35x+140}{x+2}+28$;

or $63+\frac{72}{x-1}=35+\frac{70}{x+2}+28$;

$\therefore 36(x+2)=35(x-1)$.

24. $17-\frac{60x+170}{3x-10}=\frac{6x+17}{-2x+1}$;

or $17-\left(20+\frac{370}{3x-10}\right)=-3+\frac{20}{1-2x}$;

$\therefore 37(2x-1)=2(3x-10)$.

25. $11+\frac{60x+55}{6x+5}=\frac{84x+77}{4x+7}$;

or $11+10+\frac{5}{6x+5}=21-\frac{70}{4x+7}$;

$\therefore\ -14(6x+5)=4x+7.$

26. Mult. by $\frac{1}{2}^2$: then

$3x-\frac{4(2x-3)-3(3x-1)}{x-1}=\frac{9x^2-3x+18}{3x-2}$;

or $3x+\frac{x+9}{x-1}=\frac{9x^2-3x-2}{3x-2}+\frac{20}{3x-2}$;

or $3x+1+\frac{10}{x-1}=3x+1+\frac{20}{3x-2}$;

$\therefore\ 3x-2=2(x-1)$; hence $3x=2x$,

which can be true only when $x=0$.

27. Mult. by 60 : then

$42x+39+55x-\frac{5}{2}(x-1\frac{1}{2})$

$=36x+12+\frac{1}{11}(1290x-45+120x)$;

$\therefore\ 61x+27=\frac{5}{4}(2x-3)+\frac{1}{11}(1410x-45)$;

mult. by 4 : then

$244x+108=10x-15+\frac{1}{11}(5640x-180)$;

$\therefore\ 234x+123=\frac{1}{11}(5640x-180)$;

or $78x+41=\frac{1}{11}(1880x-60)$;

mult. by 11 : then $858x+451=1880x-60$.

28. $\frac{18x-22}{3(13-2x)}+2x+\frac{2x}{3}=\frac{106-101-1}{24}+\frac{8x}{3}$;

mult. by 3 : then $\frac{18x-22}{13-2x}+6x+2x=\frac{1}{2}+8x$;

whence $2(18x-22)=13-2x$.

29. $4x-\frac{x}{2}+1-2x+\frac{1}{4}x-\frac{1}{18}\left(16-\frac{1}{2}x-2\right)=\frac{3x}{2}+3$;

mult. by 4: then

$16x-2x+4-8x+x-\frac{1}{9}(28-x)=6x+12.$

30. Mult. by 5: then

$$\frac{6-5x}{3}-\frac{35-10x^2}{14(x-1)}=\frac{5+15x+1}{21}-\frac{10x-11}{6};$$

mult. by 84: then

$$168-140x-\frac{210-60x^2}{x-1}=24+60x-140x+754;$$

or $-10-60x=\frac{210-60x^2}{x-1}$; or $6x+1=\frac{6x^2-21}{x-1}$;

$\therefore 6x^2-5x-1=6x^2-21$; whence $5x=20$.

31. $\frac{-10x+16}{2x-1}+\frac{8x+6}{x+3}=3$; or $-5+\frac{11}{2x-1}+8-\frac{18}{x+3}=3$;

$\therefore 11(x+3)=18(2x-1)$.

32. Mult. by $\frac{7}{2}$: then $\frac{42x-7}{7x+2}+\frac{13}{2}+\frac{24-24x}{7+12x}=\frac{21}{2}$;

or $\frac{42x-7}{7x+2}-\frac{24x-24}{12x+7}=4$;

or $6-\frac{19}{7x+2}-\left(2-\frac{38}{12x+7}\right)=4$;

$\therefore 12x+7=2(7x+2)$.

33. Mult. by 6: then $\frac{4x-4}{2x+1}=\frac{3x-9}{x+1}-1$;

or $2-\frac{6}{2x+1}=3-\frac{12}{x+1}-1$; $\therefore 2(2x+1)=x+1$.

34. Mult. by 4: then $\frac{16x+75}{4x+5}=\frac{15x+32}{5x-4}+1$;

or $4+\frac{55}{4x+5}=3+\frac{44}{5x-4}+1$; $\therefore 5(5x-4)=(4x+5)$.

Ex. 45.

1. If the harness cost x pounds, then the horse $2x$, and the chaise $3x \times 1\frac{1}{2}$; $\therefore\ x+2x+4\frac{1}{2}x=60$.

2. If $2x$=no. of cav., then $25x$=no. of inf., and x=no. of art.; $\therefore\ 28x=2744$.

3. If x pounds were given to the charity, then $12x$ to eldest son, $12x+1\frac{1}{2}=8x$ to each younger son, $12x+1\frac{2}{3}=7\frac{1}{2}x$ to each daughter; $\therefore\ x+12x+16x+36x=13000$.

4. If $7x$ acres are given to A, then $11x$ to B, and $18x \times 1\frac{1}{2}$ to C; $\therefore\ 7x+11x+27x=270$.

5. If he went x mi. by water, then $\frac{4}{7}x$ on foot, and $\frac{2}{5}x$ on horseback; $\therefore\ x+\frac{4}{7}x+\frac{2}{5}x=3036$;
hence $35x+20x+14x=3036 \times 35$; $\therefore\ x=44 \times 35$.

6. If the side of the square=x yds., then $(x+6)(x-4)=x^2$; whence $2x=24$; $\therefore\ x^2=12^2$.

7. If length=x yds., then breadth=$x-4$, and $x(x-4)=(x+1)(x-3)-27$; or $x^2-4x=x^2-2x-30$; whence $2x=30$.

8. If x=no. in each side of the first square, then no. of troops $=x^2+60$; $\therefore\ x^2+61=(x+5)(x-3)$; whence $x=38$, and $x^2+60=1504$.

9. If x=one no., then $150-x$=the other;
$\therefore\ \frac{x}{23}+\frac{150-x}{27}=6$; or $\frac{x}{23}+6-\frac{12+x}{27}=6$;
$\therefore\ 27x=23(12+x)$; whence $4x=276$.

10. If x=no. of hf.-crs., then $140-x$=no. of flor., and the total value in shillings is $2\frac{1}{2} \times x+(140-x) \times 2=325$; or $2\frac{1}{2}x+280-2x=325$; $\therefore \frac{1}{2}x=45$; whence $x=90$ hf.-crs.

11. If there were x lb. tea, then $1\frac{1}{3}x$ lb. coffee. No. of pence gained on tea=$4 \times x$, and on coffee=$2 \times 1\frac{1}{3}x$;
$\therefore\ 4x+2\frac{2}{3}x=600$; whence $20x=1800$.

12. If it held $5x$ gall., then $5x-x-10=\frac{3}{5}$ of $5x$;
$\therefore\ 12x-30=10x$.

13. If A had x pence at first, he gave away $(\frac{3}{28}+\frac{3}{5}$ of $\frac{3}{28})x$, or $\frac{3+15}{28}x$; $\therefore\ x-\frac{9}{14}x=20$; hence $5x=280$.

14. If $6x$ pounds=his income, then $6x-(2x-10)=3x+45$; or $4x+10=3x+45$.

15. If he had x oranges at first, then 1st remainder$=\frac{1}{2}x-1$; but 2nd remainder$=\frac{1}{2}(\frac{1}{2}x-1)-2=3$;
$\therefore\ \frac{1}{4}x-\frac{1}{2}=5$; whence $x-2=20$.

16. If B had $3x$ shill., he gave A $2x$, and then had left $x=\frac{2}{5}(18+2x)$; $\therefore\ 5x=36+4x$; whence $3x=108$.

17. If at first A had $2x$ pounds, B had $3x$; but after the payments A had $\frac{2}{3}x+1$, and B $\frac{3}{8}x-1$;
hence $\frac{2}{3}x+1=\frac{3}{4}x-2$; $\therefore\ 8x+12=9x-24$.

18. If lead and tin each$=x$ lb., then lead$+$tin$=2x$ lb., which being 5 lb. more than half the whole, $\therefore$ the whole $=4x-10$, and the copper$=\frac{1}{2}(4x-10)-5$, or $2x-10$;
$\therefore\ x=\frac{1}{3}$ of $2x+5$; or $3x=2x+15$.

19. If x pounds=the real value, then the loss$=x-42$; but the supposed gain$=52\frac{1}{2}-x=\frac{3}{4}(x-42)$; $\therefore\ 210-4x=3x-126$.

20. If he won x games at 2 shill. each, he lost $20-x$ games at 3 shill. each; $\therefore\ 2x-3(20-x)=5$, or $2x+3x=60+5$.

21. B overspends yearly $\frac{1}{4}$ of £220, or £55; $\therefore$ if he had spent only £$(80-55)$ or £25 more than A, he would have spent simply his whole income. In that case, if $5x$ pounds=the income of each, then A would spend $4x$ and B $5x$; $\therefore\ 5x-4x=25$.

22. If A saves x pence per day, B saves $x-6$; then A's whole saving$=50x$, and B's$=50(x-6)$;
$\therefore\ 50x=150(x-6)+60\times 2\frac{1}{2}$; or $2x=6(x-6)+6$.

23. If weight of cover$=x$ ounces, then 2nd cup$=\frac{1}{2}(12+x)$, and 2nd with cover$=\frac{1}{2}(12+3x)=12\times 1\frac{1}{3}$;
$\therefore\ 12+3x=32$.

24. $3x$ tickets at 6 shill.$=18x$ shill.; $\therefore$ value of watch $=18x-80$; again, $4x$ tickets at 5 shill.$=20x$ shill.; $\therefore$ value of watch$=20x-4x$, or $16x$;
hence $18x-80=16x$, or $2x=80$, or $16x=640$ shill.

25. If A has $3x$ pounds, B has $3x+1\frac{1}{2}$, or $2x$, B and C have $3x+1\frac{1}{3}$, or $2\frac{1}{4}x$; $\therefore$ C has $2\frac{1}{4}x-2x$, or $\frac{1}{4}x$; also D has $(3x+\frac{1}{4}x)\times 1\frac{1}{4}$, or $\frac{65}{16}x$;
hence $3x+2\frac{1}{4}x+\frac{65}{16}x=149$; $\therefore$ $149x=149\times 16$, or $x=16$.

26. Assume x=his no. of horses; then the no. for which he had room $=x-8$; $\therefore$ $1\frac{1}{2}(x-8)=x+8$, or $3x-24=2x+16$.

27. If $x=$ no. of widows, then the sum in shillings is $4\frac{1}{2}x+1=5x-10$; $\therefore$ $\frac{1}{2}x=11$.

28. If $3x$=no. of feet in circuit, then $3x$ palisades are required at a foot distance, but only x palisades at 3 feet distance; $\therefore$ $3x-150=x+70$; hence $x=110$ ft., and $3x=110$ yds.

29. If x=the no. of tens, then $x-4$=the no. of units, and the value of the number is $10x+(x-4)$, or $11x-4$, and the sum of the digits is $x+(x-4)$, or $2x-4$; hence $11x-4=7(2x-4)$.

30. Let x=the tens digit, then $x-2$=the units digit; $\therefore$ the value of the number $=10x+(x-2)$, or $11x-2$, and that of the number formed by inverting the digits $=10(x-2)+x$, or $11x-20$;
hence $11x-2-1\frac{1}{2}(2x-2)=11x-20$; $\therefore$ $3x-3=18$.

31. Denote the hundreds, tens, and units digits, respectively, by $x+1$, x, $x-1$; then, sum of digits$=3x$, and the value of the number$=100(x+1)+10x+(x-1)$; or $111x+99$; also that of the inverted form$=100(x-1)+10x+(x+1)$, or $111x-99$; hence
$111x+99-\frac{1}{4}(111x-99)=3x\times 36$;
$\therefore$ $(3x+99)\times 4=111x-99$.

32. Let x=no. of days required: in one day A does $\frac{1}{10}$ of the work, and B $\frac{1}{8}$; and when A has done $\frac{3}{10}$ there remain $\frac{7}{10}$; hence $\frac{x}{10}+\frac{x}{8}=\frac{7}{10}$; or $4x+5x=28$.

33. Let x=no. of days required: in one day A does $\frac{1}{10}$ of the work, B $\frac{1}{x}$; $\therefore \frac{1}{10}+\frac{1}{x}=\frac{1}{7}$; or $7x+70=10x$.

34. Let x=no. of hours required: in one hour the father reaps $\frac{1}{20}$ of the field, the son $\frac{1}{x}$; $\therefore \frac{16}{20}+\frac{6}{x}=$ the whole work, $=1$; hence $20x=16x+120$, or $5x=4x+30$.

35. Let x days denote B's time: in one day A builds $\frac{1}{8}$ of the wall, B $\frac{1}{x}$; $\therefore \frac{1}{8}+\frac{1}{x}=\frac{1}{5}$; hence $5x+40=8x$, and $x=13\frac{1}{3}$ days. Accordingly, half the wall could be built by A in 4 days, or by B in $6\frac{2}{3}$ days; so that A should begin $2\frac{2}{3}$ days after B.

36. A could do the whole work in $\frac{2}{5}$ of 10 days, =18 days. Then, if B could do the whole in x days, we have $\frac{3}{18}+\frac{3}{x}=\frac{4}{9}$; $\therefore \frac{3}{x}=\frac{5}{18}$, and $x=10\frac{4}{5}$ days.

37. If x=no. of minutes required, then, since the supply pipes fill $\frac{1}{24}+\frac{1}{30}$ of the cistern per minute, and the discharge pipe empties $\frac{1}{20}$, we have $\frac{x}{24}+\frac{x}{30}-\frac{x}{20}=1$;

or $5x+4x-6x=120$.

38. If one package weigh x lb., the other weighs $224-x$; and we have $\frac{9}{112}x+\frac{13}{112}(224-x)=24$;

$\therefore 9x+224\times13-13x=24\times112$; or $26-\frac{1}{28}x=24$.

39. If A could fill it in x minutes, then A supplies $\frac{1}{x}$ of the whole per minute, and $B\ \frac{1}{15}-\frac{1}{x}$, or $\frac{x-15}{15x}$; so that B could fill it in $\frac{15x}{x-15}$ minutes. But $\frac{5}{x}+\frac{13}{15}=1$; hence $x=37\frac{1}{2}$ minutes by A; $\therefore\ \frac{15x}{x-15}=25$ minutes by B.

40. Denote the sums by x and $330-x$; then the amount of returns is $\frac{11}{100}\frac{5}{0}x+\frac{9}{100}\frac{2}{0}(330-x)=345$;
$\therefore\ 5x+4(330-x)=1500$.

41. Suppose $2x$ gall. daily for the woman, then $3x$ for the man, $5x$ for both; $\therefore\ 5x\times 20=100x$ gall. in the full cask; hence $100x-\frac{1}{3}\frac{8}{0}=5x\times 18+2x\times 2$;
or $x=\frac{3}{2}\frac{3}{5}$; $\therefore\ 100x=12$ gall.

42. Assume x pounds for the less sum:
$\therefore\ (607\frac{1}{12}-\ x)\times\frac{7}{100}-x\times\frac{13}{160}=18\frac{4}{5}$;
or $(7285\ -12x)\times\frac{7}{100}-x\times\frac{3}{4}\frac{9}{0}=\frac{11}{5}\frac{2}{9}$;
or $(7285\ -12x)\times 14\ -x\times 195=1128\times 40$;
whence $x=156\frac{3}{5}$.

43. Suppose the child could reap per hour $\frac{6}{x}$ of the whole then the woman $\frac{10}{x}$, and the man $\frac{15}{x}$; hence $\frac{31}{x}=\frac{1}{30}$,
or $x=930$; $\therefore$ time required for the man $\frac{x}{15}=62$ hrs., &c.

44. The interests on 100 are $\frac{3}{5}$ of $3\frac{1}{2}$, and $\frac{7}{12}$ of $4\frac{1}{2}$; or $2\frac{1}{3}$ and $2\frac{5}{8}$; so that $2\frac{1}{3}$ is the discount on $102\frac{1}{3}$, and $2\frac{5}{8}$ the discount on $102\frac{5}{8}$; or the proportions of discount are $\frac{7}{3}\frac{}{0}\frac{}{7}$ and $\frac{2}{8}\frac{1}{2}\frac{}{1}$. Hence, if we denote the greater sum by x, and the less by $x-203\frac{1}{3}$, we have
$\frac{7}{3}\frac{}{0}\frac{}{7}x+\frac{2}{8}\frac{1}{2}\frac{}{1}(x-203\frac{1}{3})=24\frac{1}{2}$;
or $(\frac{2}{3}\frac{}{0}\frac{}{7}+\frac{6}{8}\frac{}{2}\frac{}{1})x=7+\frac{407\times 3}{821}\frac{}{}\frac{}{}\frac{4}{8}\frac{9}{2}\frac{6}{1}\frac{3}{}$;
or $(821+921)x=3484\times 307$;
whence $x=2\times 307$.

45. If B could reap the whole in x hours, we have $\frac{1}{20}+\frac{1}{x}=\frac{1}{12}$, whence $x=30$ hours; but B does $\frac{1}{12}-\frac{1}{16}$, or $\frac{1}{48}$, of the whole per hour more than C;

$\therefore$ C does $\frac{1}{30}-\frac{1}{48}$, or $\frac{1}{80}$, per hour, or the whole in 80 hrs.

Hence (i) B and C, together, do $\frac{1}{30}+\frac{1}{80}=\frac{11}{240}$ per hour, or the whole in $21\frac{9}{11}$ hrs.;

(ii) A, B, and C, together, $\frac{1}{20}+\frac{11}{240}=\frac{23}{240}$ per hour, or the whole in $10\frac{10}{23}$ hrs.

46. Call the hour hand H, the minute hand M, which travels 12 times as fast as H.

Suppose M has travelled x minute spaces, then H has travelled $\frac{1}{12}x$; and (i) we have $x-\frac{1}{12}x=60$; whence $x=65\frac{5}{11}$ min. past 6, or $5\frac{5}{11}$ min. past 7;

(ii) we have $x-\frac{1}{12}x=15$; whence $x=16\frac{4}{11}$ min. past 6.

47. If $10x$ shill. denote A's money, then A's family costs x shill. a day, and B's $10x\div15$, or $\frac{2}{3}x$; B's money $=\frac{2}{3}x\times12$, or $8x$; hence $8x=7x+2\frac{1}{2}$; whence $x=2\frac{1}{2}$.

48. If $8x-8=$the no. of ducks, then $7x-8=$the no. of geese;

$\therefore \frac{8x-16}{7x-16}=\frac{7}{6}$; or $49x-112=48x-96$.

49. If the tin$=x$ lb., the copper$=x+40$;

$x+40+1\frac{8}{9}x=\frac{11}{3}x$; $\therefore$ $x=36$ lb. tin;

hence $x+(x+40)=112$ lb.

50. If the weight of gold in air be x lb., that of silver is $106-x$;

$\therefore \frac{18}{19}x+\frac{9}{10}(106-x)=99$; or $\frac{2}{19}x+\frac{1}{10}(106-x)=11$;

hence $20x+19(106-x)=190\times11$;

$\therefore$ $x=19(110-106)=76$.

51. If $5x$ shill.=original value of each coin, then $4x$=its reduced value. Now, since x shill. was gained on each coin passed, ∴ the no. of coins passed=$52\frac{1}{2}\div x$, or $105+2x$, and we have

the original no. of coins=$\frac{105}{2x}\div\frac{3}{5}$, or $\frac{175}{2x}$;

also, no. of coins seized=$\frac{175}{2x}-\frac{105}{2x}-1$,

or $\frac{35}{x}-1$, each worth $4x$ shill.

Loss on the whole, $4x\left(\frac{35}{x}-1\right)-52\frac{1}{2}=\frac{4}{5}$ of $52\frac{1}{2}$;

∴ $140-4x=\frac{7}{3}$ of $52\frac{1}{2}$; or $840-24x=7\times105$.

52. Let x=whole no. of light sovereigns, equal in weight to $x-9$ true sovereigns; ∴ weight of a light sovereign $=\frac{x-9}{x}$ of a true one;

hence $\frac{x-9}{x}=\frac{20}{21}$, or $21x-189=20x$.

53. Suppose B ran for x minutes, then A ran for $x+\frac{1}{2}$ minutes. Now, B ran $(900-75)\div5$ or 165 yds. per minute;

∴ the length of the course=$165x$ yds.

A ran $900\div5$ or 180 yds. per minute for 5 minutes, and 160 yds. per minute for $x+\frac{1}{2}-5$ minutes;

∴ $900+160(x-4\frac{1}{2})=165x$; or $5x=900-720$.

54. If A had travelled for x hours when he was overtaken, B had then travelled for $x-2$ hours; hence $4x=5(x-2)$;

∴ $x=10$. Accordingly, the whole distance is $4x=40$ miles, and C's whole time is $40\div6=6\frac{2}{3}$ hours; but B's time is $x-2=8$ hours; ∴ C must start $1\frac{1}{3}$ hour after B.

55. If the investment in the $3\frac{1}{2}$ per Cents$=x-100$, then that in the $4\frac{1}{2}$ per Cents$=x+100$, and the whole money invested$=2x$.

The joint income is $(x-100)\times\frac{3\frac{1}{2}}{85}+(x+100)\times\frac{4\frac{1}{2}}{108}$

$=2x\times\frac{4\frac{1}{7}}{100}$;

hence $\frac{7}{170}x+\frac{1}{24}x-\frac{29}{350}x=\frac{700}{170}-\frac{100}{24}=-\frac{5}{102}$;
mult. by 170: then
$7x+\frac{85}{12}x-\frac{493}{35}x=-\frac{25}{3}$;
mult. by 420: then $(2940+2975-5916)x=-3500$.

Ex. 46.

1. $4x+18y=22$,
$4x+\ y=\ 5$,
$\therefore\ 17y=17$.

2. $y=a-x$,
$\therefore\ ax+b(a-x)=b^2$,
or $(a-b)x=-b(a-b)$.

3. $y=2x-8$,
$\therefore\ 2(2x-8)+x=9$,
or $5x=25$.

4. $y=b-ax$,
$\therefore\ x+b(b-ax)=a$,
$(1-ab)x=a-b^2$.

5. $8x-36y=44$,
$9x-36y=45$,
$\therefore\ x=1$,
and $2-9y=11$.

6. $b^2x+aby=b^2$,
$a^2x-aby=a^2$,
$\therefore\ (a^2+b^2)x=a^2+b^2$,
and $a^2-aby=a^2$.

7. $2x+3y=\ 8$,
$21x-3y=15$,
$23x\ \ =23$,
and $y=7x-5$.

8. $ax-by=\ 0$,
$bx+by=bc$,
$\therefore\ (a+b)x=bc$,
and $y=ax\div b$.

9. $35x+28y=406$,
$12x+28y=268$,
$\therefore\ 23x=138$,
and $30+4y=58$.

10. $7x=y$,
$2x+20=21x+1$,
$\therefore\ x=1$,
and $y=7x=7$.

11. $3x+2y=78$,
$3\frac{1}{2}x+2y=80$,

$\therefore \frac{1}{2}x=2$,
and $30+2y=78$.

12. $8x+9y=43\times72$,
$9x+8y=42\times72$,

$\therefore x-y=-72$,
$8x-8y=-78\times8$,
add (ii) $17x=(42-8)\times72$.

13. By addition:
$46x-46y=92$,
$\therefore x-y=2$,
$13x-13y=26$,
$21x-13y=90$,

$\therefore 8x=64$,
and $y=x-2$.

14. By subtraction:
$2x-2y=18$,
$\therefore 30x-30y=270$,
$17x+30y=59$,

$\therefore 47x=329$,
and $y=x-9$.

15. By subtraction:
$30x-60y=0$,
or $x-2y=0$,
or $x=2y$,
$\therefore 32y+11y=86$.

16. $\frac{b}{a}x-y=bm$,
$\frac{d}{c}x+y=dn$,

$\therefore \frac{ad+bc}{ac}x=bm+dn$.

17. $10x-y=17$, and $x+9y=29$,
$90x-9y=153$,

$\therefore 91x=182$, and $y=10x-17$.

18. From (ii), $\frac{a}{c}x=\frac{b}{a}y$; whence $y=\frac{a^2}{bc}x$;

By (i), $\frac{x}{b}+\frac{a^2}{bc^2}x=1$; $\therefore x\left(1+\frac{a^2}{c^2}\right)=b$;

$\therefore x=\frac{bc^2}{a^2+c^2}$; and $y=\frac{a^2}{bc}\times\frac{bc^2}{a^2+c^2}$.

19. From (ii), $by-ax=a^2-b^2$,
(i), $by+ax=c^2$;

by subtraction:
$2ax=c^2-(a^2-b^2)$;
by addition:
$2by=c^2+(a^2-b^2)$.

20. Mult. (i) by ab^2, (ii) by a^2b:
$b^2x+aby=ab^2$,
$a^2x-aby=a^2b$,
$\therefore (a^2+b^2)x=ab(a+b)$.

Mult. (i) by a^2b, (ii) by ab^2:
$\therefore (a^2+b^2)y=ab(a-b)$.

21. $\frac{x}{a}+\frac{x}{c}+\frac{y}{b}=1$, and $\frac{y}{a}-\frac{y}{c}+\frac{x}{b}=1$;

$\therefore x\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\right)=y\left(\frac{1}{a}-\frac{1}{b}-\frac{1}{c}\right)$;

hence $y=x\cdot\frac{bc-ac+ab}{bc-ac-ab}=x\left(1+\frac{2ab}{bc-ac-ab}\right)$;

and (i) becomes $x\left(\frac{1}{a}+\frac{1}{c}+\frac{1}{b}+\frac{2a}{bc-ac-ab}\right)=1$;

or $x\left\{\frac{bc+(ab+ac)}{abc}+\frac{2a}{bc-(ab+ac)}\right\}=1$;

or $x\left\{\frac{b^2c^2-(ab+ac)^2+2a^2bc}{abc(bc-ab-ac)}\right\}=1$;

whence x=the reciprocal of the above coefficient.

Again: $x=y\cdot\frac{bc-ac-ab}{bc-ac+ab}=y\left(1-\frac{2ab}{bc-ac+ab}\right)$;

and (ii) becomes $y\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}-\frac{2a}{bc-ac+ab}\right)=1$;

or $y\left\{\frac{bc+(ac-ab)}{abc}-\frac{2a}{bc-(ac-ab)}\right\}=1$;

&c. &c.

22. Mult. (i) by 6, and (ii) by 2: which will give $4x+3y=48$, and $-3x+5y=22$.

23. Mult. (i) by 20, and (ii) by 6: which will give $6x-5y=8$, and $4x+15y=42$.

24. Mult. 1st and 2nd expressions by 7: which gives $30x+26y=49x-98y+49$;
or $-19x+124y=49$. (i)

Mult. 2nd and 3rd expressions by 6: which gives $42x-84y+42=12\{6y-18+2(2y+15)\}$;
or $21x-102y=51$. (ii)

By addition: $2x+22y=100$, or $x+11y=50$;
hence $19x+209y=950$, to which add (i); &c.

25. $7x-y+2=35$; or $7x-y=33$;
$12y-x-10=9$; or $-x+12y=19$.

26. $106y+9x-54y+9=7x-21$; or $x+26y=-15$;
$7x-35y+56=27x-117y+55$; or $-20x+82y=-1$.

27. Mult. (ii) by 12: $9y+5x-8-3x-3y=\frac{12}{11}(7x+6)$;
or $x+3y-4=\frac{6}{11}(7x+6)$;

mult. by 11: $11x+33y-44=42x+36$;
or $-31x+33y=80$;

(i) $\times 30$ gives $9x+12y+9-6x+2y=150+6y-48$;
or $3x+8y=93$.

28. $40x-5y-15=140+12y-8x$; or $48x-17y=155$;
$24y-16+2x=147-6y-3$; or $x+15y=80$.

29. $x-\frac{x-2y}{x-23}=20-\frac{5.9}{y}+x$; or $\frac{x-2y}{x-23}=\frac{1.9}{y}$;

or $19x-437=2x-4y$; or $17x+4y=437$; (i)

$y+\frac{y-3}{x-18}=30-\frac{7.3}{y}+y$; or $\frac{y-3}{x-18}=\frac{1.7}{y}$;

or $17x-306=3y-9$; or $17x-3y=297$. (ii)

Ex. 47.

1. (iii)$-$(ii): $y+z=5$; (ii)$-$(i): $x+y+z=6$; $\therefore x=1$.
By (i), $3y+4z=18$; $\therefore 3y+4(5-y)=18$; $\therefore y=2$.

2. (ii)$\times 4$: $8y-4z=44$,
(iii) $3x+4z=57$,
$\therefore 3x+8y=101$;
(i) $5x+3y=65$.

3. (i)$-$(ii): $x-y-7z=-50$,
(iii) $x-y+6z=41$,
$\therefore 13z=91$; or $z=7$;
by (i): $3x+2y=27$,
by (iii): $2x-2y=-2$.

4. (i)$+$(iii): $2x=8$; $\therefore x=4$;
(i)$+$(ii): $2x+2y=-2$; or $8+2y=-2$; &c.

5. (i) $7-x=2y$,
(ii) $4-4z=2y$; $\therefore x=4z+3=7-2y$;
(iii) $z-3x-1=2y$; $\therefore z-3(4z+3)-1=4-4z$; &c.

6. (i)$-$(ii): $y-x=a-b$; (iii) $y+x=c$;
$\therefore 2x=c+b-a$; and $2y=c-b+a$.

7. These equations may be written

$$1=\frac{1}{y}+\frac{1}{x},\ 1=\frac{2}{z}+\frac{2}{x},\ 1=\frac{3}{z}+\frac{3}{y};$$

hence (i) $\frac{6}{y}+\frac{6}{x}=6$; (ii) $\frac{6}{z}+\frac{6}{x}=3$; (iii) $\frac{6}{z}+\frac{6}{y}=2$.

By addition we have $\frac{12}{x}+\frac{12}{y}+\frac{12}{z}=11$; from which subtract, separately, twice (iii), twice (ii), twice (i).

8. (i) $2x-2y-3z=-2$,
(ii) $x-3y-3z=-1$,
(iii) $2x+4y+3z=4$;

(i)$-$(ii): $x+y=-1$,
(ii)$+$(iii): $3x+y=3$.

9. Mult. the equations respectively by 12, 6, and 6.

(i) $6x+4y+2z=144$,
(ii) $-x+3y+2z=48$,
(iii) $3x+2z=60$;

(i)$-$(ii): $7x+y=96$,
(iii)$-$(ii): $4x-3y=12$.

10. $-\frac{1}{8}x+y+\frac{1}{3}z=5$,
$\frac{1}{4}x-\frac{1}{8}y-\frac{1}{10}z=\frac{3}{20}$,
$x-\frac{2}{3}y+\frac{1}{12}z=\frac{1}{12}$.

Mult. these respectively by 30, 100, and 120; then add (i) and (ii), also (ii) and (iii); whence $19x+10y=165$, and $145x-100y=25$.

Ex. 48.

1. Let x=no. of shillings for ale, y=ditto for stout.
   (i) $7x+17y=267$; (ii) $12x+7y=192$;
   subtract, and divide by 5.

2. Let x=no. of pounds for a horse, y=ditto for a cow.
   (i) $9x+7y=300$; (ii) $6x+13y=300$;
   subtract, and divide by 3.

3. Take x yds. and y yards for the lengths;
   (i) $8x+9y=253$; (ii) $2x+2y=60$, whence $y=30-x$.

4. Take x lb. of tea sold at a *loss* of 3*d.* per lb., and y lb. of coffee sold at a *profit* of 4*d.* per lb.
   (i) $45x+14y=3060$; (ii) $4y-3x=240$;
   multiply (ii) by 15.

5. Suppose x men, y women.
   (i) $3x+y=46$; (ii) $2\frac{1}{2}x+1\frac{1}{2}y=45$;
   mult. (i) by 3, and (ii) by 2.

6. Suppose $12x$ men, $12y$ women, $12z$ children; then, expressing the money in shillings, we have
   (i) $30x+12y+12z=258$; (ii) $12x+20y+12z=258$;
   (iii) $12x+12y+15z=258$.
   (i) $-$(ii) gives $8y=18x$, or $12y=27x$;
   (ii)$-$(iii) gives $8y=3z$, or $12z=72x$;
   $\therefore 30x+27x+72x=258$; whence $x=2$;
   and $12x+27x+72x=222$ persons.

7. Take x for numerator, y for denominator.

$\frac{x+7}{y}=\frac{3}{5}$; $\therefore 3x-2y=-21$; (i)

$\frac{x}{y-7}=\frac{3}{5}$; $\therefore 8x=3y=-21$. (ii)

8. Suppose x crowns, y half-guineas; then, expressing the values in guineas, we have

$\frac{5}{21}x+\frac{1}{2}y=25$, or $10x+21y=1050$; (i)
and $2y-3x=17$. (ii)

9. Suppose A has x shill. per day, B y shill.
(i) $15x+14y=117$; (ii) $4x-3y=11$.

10. Suppose coffee at x pence, sugar at y pence.
$3x+5y+x+2\frac{1}{2}=120$, or $8x+10y=235$; (i)
$2\frac{1}{2}x+8y+y+2\frac{1}{2}=120$, or $5x+18y=235$. (ii)

11. Let $A=x$, $B=y$, $C=z$; then (i) $x+\frac{1}{2}y=1000$;
(ii) $y+\frac{1}{3}z=1000$; (iii) $\frac{1}{4}x+z=1000$.
(i) $\times 6-$(ii) $\times 3$ gives $6x-z=3000$; to which add (iii).

12. Take x apples at $\frac{1}{4}d.$, and y pears at $\frac{1}{5}d.$
(i) $\frac{1}{4}x+\frac{1}{5}y=30$; (ii) $\frac{1}{8}x+\frac{1}{15}y=13$.
(i) $\times$ 5 gives $\frac{5}{4}x+y=150$
(ii) $\times 15$ gives $\frac{15}{8}x+y=195$
$\therefore \frac{5}{8}x=45$.

13. Let x ft.=length, y ft.=breadth; then xy=area.
$(x+4)(y+5)=xy+116$; whence $5x+4y=96$; (i)
$(x+5)(y+4)=xy+113$; whence $4x+5y=93$; (ii)
adding &c. gives $x+y=21$, subtracting: $x-y=3$.

14. Let x=price of the 3 per cents, y=that of the $3\frac{1}{2}$.

$\frac{2000}{x}$ cents at £3 $+\frac{3300}{y}$ cents at £$3\frac{1}{2}$=£$211\frac{1}{4}$;

$\frac{550}{x}$ cents at £3 $-\frac{550}{y}$ cents at £$3\frac{1}{2}$=£$\frac{1}{8}$.

$\therefore$ (i) $\frac{24000}{x}+\frac{46200}{y}=845$ } Mult. (ii) by 3,
(ii) $\frac{13200}{x}-\frac{15400}{y}=1$ } and then add (i).

15. Suppose x men, y women.
$\therefore x=1\frac{1}{2}(y-4)$, or $2x-3y=-12$; (i)
$x+4=1\frac{1}{3}y$, or $3x-4y=-12$. (ii)

16. Let x=the tens digit, y the units; $\therefore$ the value of the number$=10x+y$.
$10x+y-3=6(x+y)$, or $4x-5y=3$; (i)
whence it appears that x is the greater digit;
$x+y=6(x-y)$, or $5x-7y=0$. (ii)

17. Denoting the tens digit by x, and the units by y, the value of the number$=10x+y$, and that of the number inverted$=10y+x$; hence
$\frac{10x+y}{x+y}=4$, or $6x-3y=0$, or $2x-y=0$; (i)
$\frac{10y+x+12}{x+y}=8$, or $7x-2y=12$. (ii)

18. Denoting the hundreds by x, the tens and units each by y, we have
(i) $(100y+10y+x)-2(100x+10y+y)=42$;
(ii) $(100y+10y+x)-(100y+10x+y)=27$.
From (i): $88y-199x=42$; from (ii): $y=x+3$;
$\therefore 88(x+3)-199x=42$.

19. Suppose, at first, $3x$ women at $2y$ pence; $\therefore 4x$ men at $3y$ pence; and the whole reckoning$=6xy+12xy=18xy$.
Now $(4x-1)$ men at $3d.+(3x-1)$ women at $2d.=85d.$
$\therefore 12x-3+6x-2=85$; whence $x=5$;
$\therefore 18xy=90y$ pence, the reckoning;
$(3y+3)\times 19+(2y+2)\times 14=90y$; whence $y=17$.

20. Let x and $80-x$ denote the parts of 80, and let y and $90-y$ denote those of 90.
$\therefore$ (i) $x+y=100$; whence $y=100-x$.
(ii) $(80-x)-(90-y)=30$;
$\therefore$ $(80-x)-(90-100+x)=30$; whence $x=30$; $80-x=50$:
or (ii) $(90-y)-(80-x)=30$;
$\therefore$ $(90-100+x)-(80-x)=30$; whence $x=60$; $80-x=20$.

21. Let x=number of games A won at $2s.$ each, y=number he lost at $3s.$ each; $\therefore$ $2x-3y=3$. (i)
In the *2nd* case: $5(y+1)-2(x-1)=30$;
$\therefore$ $2x-5y=-23$. (ii)

22. Denote the digits by x, y, and $2x$; the value of the number is $100x+10y+2x$, or $102x+10y$.
$\therefore$ (i) $(102x+10y)+(3x+y)=22$; or $y=3x$;
(ii) $(102x+10y)+2xy=11$; or $51x+5y=11xy$.
(ii) by substitution becomes $51x+15x=33x^2$; $\therefore$ $x=2$.

23. Let x shillings=the 1st-class fare, $\therefore$ $\frac{1}{8}(4x-3)$=the 2nd-class fare; also, if y=the original no. of 1st-class passengers, $y+40$=that of 2nd ditto. Now, rating the additional passengers at half-price, we have
$5x+\frac{1}{8}^3(4x-3)=\frac{1}{8}$ of 885,
or $25x+52x-39=885$; whence $x=12$ shill., the 1st-class fare, and $\frac{1}{8}(4x-3)=9$ shill., the 2nd ditto. Accordingly, the amount paid at first was $12y+9(y+40)=885$; whence $y=25$; and the whole no. of passengers $=2y+40=90$.

24. Suppose x measures=size of a bale, y such measures=size of a cask; $\therefore$ size of the cave $=13x+33y=3(5x+9y)$; whence $y=\frac{1}{3}x$; and, first, by substituting for y, we find that the cave could hold $13x+11x$, or 24 bales; secondly, substituting for x, we obtain $39y+33y$, or 72 casks.

25. Let x=no. of days for A by himself, y for B, z for C;
then A can do $\frac{1}{x}$ of the whole per day, $B\,\frac{1}{y}$, $C\,\frac{1}{z}$; and
we have (i) $\frac{1}{x}+\frac{1}{y}=\frac{1}{12}$, (ii) $\frac{1}{y}+\frac{2}{z}=\frac{1}{15}$,
(iii) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{10}$;
(iii)$-$(i) gives $z=60$; and (ii) becomes $\frac{1}{y}+\frac{1}{30}=\frac{1}{15}$,
$\therefore\ y=30$.

26. Suppose he could row x miles per hour in still water, and that the stream flows at the rate of y miles per hour. He goes down the stream at the rate of $x+y$ miles per hour, and up at the rate of $x-y$;
$\therefore$ (i) $\frac{20}{x+y}+\frac{20}{x-y}=10$; (ii) $\frac{1}{3}(x+y)=\frac{1}{2}(x-y)$.

$$\left.\begin{array}{l}\text{From (ii)}\ \dfrac{3}{x+y}-\dfrac{2}{x-y}=0\\ \text{from (i)}\ \dfrac{2}{x+y}+\dfrac{2}{x-y}=1\end{array}\right\}\ \therefore\ \frac{5}{x+y}=1;$$

$\therefore\ x+y=5$, and $x-y=3\frac{1}{3}$; and hence $20\div5=4$ hrs. going; $20\div3\frac{1}{3}=6$ hrs. returning.

27. Suppose x shill. per quarter, y shill. per barrel.
(i) $18x=\frac{104}{100}$ of $25y$; (ii) $16y-11x=4$.
From (i): $9x-13y=4$; &c.

28. Denoting the tens by x, and the units by y, the value of the number$=10x+y$, and that of the inverted number $=10y+x$; $\therefore\ x+y+\frac{9}{16}(10x+y)=10y+x$; whence $x=1\frac{1}{2}y$;
$\therefore$ value of no.$=16y$, difference of digits$=\frac{1}{2}y$, and product of digits$=1\frac{1}{2}y^2$; hence the fraction required $=\frac{1\frac{1}{2}y^2}{16y\times\frac{1}{2}y}=\frac{3}{16}$.

29. Suppose x pounds invested in the 4 per Cents, y pounds in the $4\frac{1}{2}$ per Cents.

Income from the first $=4 \times \frac{x}{86\frac{1}{4}}=\frac{16x}{345}$;

" " second$=4\frac{1}{2}\times\frac{y}{97\frac{7}{8}}=\frac{4y}{87}$;

hence $\frac{16x}{345}=\frac{4y}{87}$, or $116x-115y=0$. (i)

Proceeds obtained by selling out$=\frac{x}{86\frac{1}{4}}\times 90=\frac{24x}{23}$;

hence $\frac{24x}{23}-40=y$, or $24x-23y=920$. (ii)

From (ii) $\times 5$ subtract (i).

30. Suppose A had x shill. at first, and B y shill.; then, since the three had 3 times 24 shill., C had at first $72-x-y$ shill.

At the end of the 1st game, A (having lost $72-x$) had $2x-72$, B had $2y$, and C $144-2x-2y$. At the end of the 2nd game, A had $4x-144$, and B (having lost $72-2y$) had $4y-72$; which sums, by the question, were half as much as they had at the end of the 3rd game.

$\therefore$ (i) $4x-144=12$; (ii) $4y-72=12$.

Ex. 50.

1. $\left(a^{-\frac{3}{2}}b^{\frac{2}{3}}\right)^{-\frac{2}{3}}=ab^{-\frac{2}{3}}$; $a^{\frac{2}{3}}a^{-\frac{1}{6}}=a^{\frac{1}{2}}$; $a^{-\frac{1}{2}}a^{\frac{3}{4}}a^{-\frac{4}{8}}=a^{-\frac{1}{4}}$;
$\left(a^2ba^{-\frac{4}{3}}b^{-\frac{2}{3}}\right)^6=\left(a^{\frac{2}{3}}b^{\frac{1}{3}}\right)^6=a^4b^2$.

2. $\left(x^{-\frac{3}{2}}y \cdot x^{-\frac{1}{2}}y \cdot x^{\frac{2}{3}}y^{-\frac{2}{3}}\right)^3=\left(x^{-2}x^{\frac{2}{3}}y^2y^{-\frac{2}{3}}\right)^3=\left(x^{-\frac{4}{3}}y^{\frac{4}{3}}\right)^3=$&c.
$\left(x^{\frac{1}{2}}y^{-\frac{1}{4}}x^{\frac{1}{6}}y^{\frac{1}{4}}y^{1\frac{1}{2}}\right)^3=\left(x^{\frac{2}{3}}y^{1\frac{1}{2}}\right)^3=$&c.

3. $\left(x^3y^2x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{4}{3}}x^{-\frac{1}{12}}y^{-\frac{1}{6}}z^{-\frac{1}{4}}\right)^{\frac{12}{13}}=\left(x^{\frac{13}{4}}y^{\frac{13}{6}}z^{\frac{13}{12}}\right)^{\frac{12}{13}}=$ &c.;
$\sqrt[n]{a^{2m}b^{6m}c^m}=a^2b^6c$.

Ex. 53.

1. Let $\sqrt{(4+2\sqrt{3})}=\sqrt{x}+\sqrt{y}$ (i); $\therefore$ $\sqrt{(4-2\sqrt{3})}$ $=\sqrt{x}-\sqrt{y}$ (ii);
(i) squared gives $4+2\sqrt{3}=x+y+2\sqrt{xy}$; $\therefore$ $x+y=4$;
(i) $\times$ (ii) gives $\sqrt{(16-12)}=x-y$; $\therefore$ $x-y=2$;
whence $x=3$, $y=1$; and $\sqrt{x}+\sqrt{y}=\sqrt{3}+1$.

2. Let $\sqrt{(11+6\sqrt{2})}=\sqrt{x}+\sqrt{y}$ (i); $\therefore$ $\sqrt{(11-6\sqrt{2})}$ $=\sqrt{x}-\sqrt{y}$ (ii);
(i) squared gives $11+6\sqrt{2}=x+y+2\sqrt{xy}$; $\therefore$ $x+y=11$;
(i) $\times$ (ii) gives $\sqrt{(121-72)}=x-y$; $\therefore$ $x-y=7$; &c.

3. Let $\sqrt{(8-2\sqrt{15})}=\sqrt{x}-\sqrt{y}$ (i); then $\sqrt{(8+2\sqrt{15})}$ $=\sqrt{x}+\sqrt{y}$ (ii);
(i) squared gives $8-2\sqrt{15}=x+y-2\sqrt{xy}$; $\therefore$ $x+y=8$;
(i) $\times$ (ii) gives $\sqrt{(64-60)}=x-y$; $\therefore$ $x-y=2$; &c.

4. Let $\sqrt{(38-12\sqrt{10})}=\sqrt{x}-\sqrt{y}$ (i). We may now proceed as in the last example; or we may determine the root as follows:
Squaring gives $38-12\sqrt{10}=x+y-2\sqrt{xy}$;
$\therefore$ $x+y=38$ (i); and $2\sqrt{xy}=12\sqrt{10}$, or $xy=360$ (ii);
subtracting 4 times (ii) from the square of (i) gives $(x-y)^2=4$; &c.

6. Root required $=\sqrt{\frac{1}{4}(9-4\sqrt{5})}=\frac{1}{2}\sqrt{(9-4\sqrt{5})}$.

7. Root required $=\sqrt{\frac{1}{9}(39-12\sqrt{3})}=$ &c.

8. Root required $=\sqrt{\frac{1}{144}(137-48\sqrt{2})}=$ &c.

9. Let $\sqrt{(17+12\sqrt{2})}=\sqrt{x}+\sqrt{y}$;
squaring gives $17+12\sqrt{2}=x+y+2\sqrt{xy}$;
$\therefore\ x+y=17$; and $2\sqrt{xy}=12\sqrt{2}$, or $xy=72$.
Put $x=8\frac{1}{2}+z$, then $y=8\frac{1}{2}-z$, and $xy=72\frac{1}{4}-z^2=72$;
$\therefore\ z=\frac{1}{2}$;
whence $x=9$, $y=8$, and $\sqrt{x}+\sqrt{y}=3+2\sqrt{2}$.
Now let $\sqrt{(3+2\sqrt{2})}=\sqrt{u}+\sqrt{v}$; $\therefore\ 3+2\sqrt{2}$
$=u+v+2\sqrt{uv}$;
whence $u+v=3$, and $2\sqrt{uv}=2\sqrt{2}$, or $uv=2$.
Put $u=1\frac{1}{2}+w$, then $v=1\frac{1}{2}-w$, and $uv=2\frac{1}{4}-w^2=2$;
$\therefore\ w=\frac{1}{2}$; whence $u=2$, $v=1$, and $\sqrt{u}+\sqrt{v}=\sqrt{2}+1$.

11. $\sqrt{\frac{1}{4}(6\sqrt{5}+14)}$ will be found$=\frac{1}{2}(3+\sqrt{5})=\frac{1}{4}(6+2\sqrt{5})$;
and the square root of this$=\frac{1}{2}(\sqrt{5}+1)$.

12. $\sqrt{\frac{1}{16}(769+184\sqrt{15})}$ will be found$=\frac{1}{4}(23+4\sqrt{15})$, and
the square root of this$=\frac{1}{2}(\sqrt{20}+\sqrt{3})=\sqrt{5}+\frac{1}{2}\sqrt{3}$.

Ex. 54.

1. Squaring gives $5x+10=5x+4\sqrt{5x}+4$; $\therefore\ 4\sqrt{5x}$
$=10-4$; &c.

2. $\sqrt{x-\frac{\sqrt{a}}{\sqrt{x}}}=\sqrt{a+x}$. Squaring, &c., gives $\frac{\sqrt{a}}{x}=\sqrt{a}+2$.

3. $\sqrt{b(a+x)}=\sqrt{x}(1-\sqrt{b})$; $\therefore\ b(a+x)=x(1+b-2\sqrt{b})$;
or $ab=x(1-2\sqrt{b})$.

4. $bx+x^2=1+2x+x^2$; $\therefore\ (b-2)x=1$.

5. $\sqrt[3]{17x-26}=3(\frac{25}{24}-\frac{3}{8})=2$; $\therefore\ 17x-26=8$.

6. $a+x-b=\sqrt{a^2+x^2}$; $\therefore\ a^2+x^2+2ax-2bx-2ab+b^2$
$=a^2+x^2$;
or $(2a-2b)x=b(2a-b)$.

7. $x-a=x+\sqrt{b+x}$; $\therefore\ b+x=a^2$; &c.

8. $\sqrt{a}+\sqrt{x}=\sqrt{x+2\sqrt{(ax+a^2)}}$;
$\therefore\ a+x+2\sqrt{ax}=x+2\sqrt{(ax+a^2)}$;
hence $a^2+4a\sqrt{ax}+4ax=4ax+4a^2$;
$\therefore\ 4\sqrt{ax}=3a$, and $ax=\frac{9}{16}a^2$; &c.

9. $(a-b)+x=\sqrt{2ax+x^2}$;
$\therefore\ (a-b)^2+2(a-b)x+x^2=2ax+x^2$; $\therefore\ 2bx=(a-b)^2$.

10. $(b-a)-x=\sqrt{a^2+bx+x^2}$;
$\therefore\ (b-a)^2-2(b-a)x+x^2=a^2+bx+x^2$;
$b^2-2ab=3bx-2ax=(3b-2a)x$.

Ex. 55.

4. $\frac{1}{1+x}+\frac{1}{1-x}=\frac{8}{3}$; or $\frac{1}{1-x^2}=\frac{4}{3}$.

5. Mult. by $12x^2$: then $9-2=28x^2$.

6. Mult. by $7x$: then $56x^2+49=65x^2$.

9. Dividing num^{rs} by den^{rs}: $3-\frac{36}{x^2+3}+4+\frac{54}{x^2+9}=7$;
$\therefore\ \frac{2}{x^2+3}=\frac{3}{x^2+9}$; or $3x^2+9=2x^2+18$.

11. $\frac{10x^2+17-10x^2+8}{18}=\frac{12x^2+2}{11x^2-8}$; $\therefore\ \frac{12x^2+2}{11x^2-2}=\frac{25}{18}$;
or $1+\frac{x^2+10}{11x^2-8}=1+\frac{7}{18}$; $77x^2-56=18x^2+180$.

12. $\frac{14x^2+16-14x^2}{21}=\frac{2x^2+8}{8x^2-11}$; or $\frac{8}{21}=\frac{x^2+4}{8x^2-11}$; &c.

13. $\frac{1}{x+\sqrt{(2-x^2)}}+\frac{1}{x-\sqrt{(2-x^2)}}=\frac{x}{2}$;
$\therefore\ \frac{2x}{2x^2-2}=\frac{x}{2}$; hence $2x^2-2=4$.

14. $\frac{2\sqrt{(a^2-x^2)}}{x^2}=\frac{a}{x^2}$; $\therefore\ a^2-x^2=\frac{1}{4}a^2$, or $x^2=\frac{a^2}{4}\times 3$; &c.

15. By (88), $\frac{\sqrt{(a^2-x^2)}}{\sqrt{(b^2+x^2)}}=\frac{c+d}{d-c}$; $\therefore \frac{a^2-x^2}{b^2+x^2}=\frac{(c+d)^2}{(c-d)^2}$;
hence $a^2(c-d)^2-b^2(c+d)^2=x^2\{(c+d)^2+(c-d)^2\}$
$=x^2(2c^2+2d^2)$.

16. $x\sqrt{(a^2+x^2)}+a^2+x^2=na^2$, or $x\sqrt{(a^2+x^2)}=(n-1)a^2-x^2$
$\therefore a^2x^2+x^4=(n-1)^2a^4-2(n-1)a^2x^2+x^4$;
hence $x^2=(n-1)^2a^2-2(n-1)x^2$,
or $(2n-1)x^2=(n-1)^2a^2$; &c.

Ex. 59.

11. $\frac{1}{3+x}+\frac{1}{3+2x}=-\frac{1}{3}$; $\therefore 18+9x=-9-9x-2x^2$;
$\therefore x^2+9x=-13\frac{1}{2}$; whence $x+\frac{9}{2}=\pm\frac{3\sqrt{3}}{2}$; &c.

12. Mult. by $6x$: then $2x^2+44x-24=27x^2-18x$.

13. $\frac{x+2}{x-1}-\frac{2}{x}=\frac{7}{3}-\frac{1}{2}$; $\therefore 6(x^2+2)=11(x^2-x)$.

14. $\frac{3}{5-x}+\frac{1}{4-x}=\frac{8}{x+2}$; $\therefore \frac{17-4x}{x^2-9x+20}=\frac{8}{x+2}$;
$9x-4x^2+34=8x^2-72x+160$; or $4x^2-27x=-42$.

15. $6(x^2+x^2+2x+1)=13(x^2+x)$; $\therefore x^2+x=6$.

16. $\frac{x-4}{2x+1}=\frac{x-6}{x-2}$; $\therefore 2x^2-11x-6=x^2-6x+8$.

17. $1-\frac{3}{x-5}+2+\frac{8}{x+4}=3+\frac{7}{x+1}$; $\therefore \frac{8}{x+4}-\frac{3}{x-5}=\frac{7}{x+1}$;
or $\frac{5x-52}{x^2-x-20}=\frac{7}{x+1}$; whence $x^2+20x^2=44$.

18. $\frac{2x+3}{2x-1}+\frac{x-7}{x+1}=\frac{6x-14}{3x-4}$;
or $1+\frac{4}{2x-1}+1-\frac{8}{x+1}=2-\frac{6}{3x-4}$;
$\therefore \frac{2}{2x-1}+\frac{3}{3x-4}=\frac{4}{x+1}$; or $\frac{12x-11}{6x^2-11x+4}=\frac{4}{x+1}$;
$\therefore 12x^2+x-11=24x^2-44x+16$; hence $4x^2-15x=-9$.

19. Mult. by $12a$: then $4a^2+15ax=4x^2$; $\therefore x^2-\frac{15a}{4}x=a^2$; &c.

20. $\frac{(3m-2a)x-2x^2}{4a-6m}=\frac{m^2-4a^2}{4a-6m}$;

$\therefore 2x^2-(3m-2a)x=4a^2-m^2$;

hence $4x^2-2(3m-2a)x+\frac{1}{4}(3m-2a)^2$
$=\frac{1}{4}(m^2-12am+36a^2)$;

$\therefore 2x-\frac{1}{2}(3m-2a)=\pm\frac{1}{2}(m-6a)$; &c.

21. $1-\frac{6}{x-3}-1+\frac{6}{x+3}=a$; $\therefore \frac{1}{x-3}+\frac{1}{x+3}=\frac{a}{6}$; $\therefore \frac{2x}{x^2-9}=\frac{a}{6}$;

hence $x^2-\frac{12}{a}x=9$; $\therefore x^2-\frac{12}{a}x+\frac{36}{a^2}=\frac{9}{a^2}(a^2+4)$; &c.

22. $x^2-\frac{m^3-n^3}{m^2n}x=\frac{1}{m}$;

$\therefore x^2-\frac{m^3-n^3}{m^2n}x+\frac{m^6-2m^3n^3+n^4}{4m^4n^2}=\frac{m^6+2m^3n^3+n^4}{4m^4n^2}$; &c.

23. $36x^2-25ab=30(a-b)x-6a^2-12ab-6b^2$;
or $36x^2-30(a-b)x=-6a^2+13ab-6b^2$;
$\therefore 36x^2-30(a-b)x+\{2\frac{1}{2}(a-b)\}^2=\frac{1}{4}a^2+\frac{1}{2}ab+\frac{1}{4}b^2$;
and hence $6x-2\frac{1}{2}(a-b)=\pm\frac{1}{2}(a+b)$.

24. $8x^2+8a^2+8ab+3bx=20ax+7bx$;
$\therefore x^2+a^2+ab=2\frac{1}{2}ax+\frac{1}{2}bx$;
or $x^2-\frac{1}{2}(5a+b)x=-a^2-ab$;
$\therefore x^2-\frac{1}{2}(5a+b)x+\frac{1}{16}(5a+b)^2=\frac{1}{16}(9a^2-6ab+b^2)$.

Ex. 60

1. $2x+6-x+1=\frac{3}{5}(x^2+2x-3)$; or $8(x+7)=3x^2+6x-9$;
or $3x^2-2x=65$; &c.

2. $48x+480-165x-495=-5x^2-65x-150$;
or $5x^2-52x=-135$; &c.

3. $14-x=\frac{63-9x}{3x-13}$; hence $3x^2-64x=-245$; &c.

4. $6x-14+\frac{8x^2-20x}{x+5}=(7x$; $\therefore\ 8x^2-20x=(x+5)(x+14)$; &c.

5. $\frac{6x}{x-4}+\frac{6x-15}{x-3}=25$; or $6+\frac{24}{x-4}+6+\frac{3}{x-3}=25$;

$\therefore\ \frac{24}{x-4}+\frac{3}{x-3}=13$; hence $13x^2-118x=-240$; &c.

6. $4x+18+\frac{72x-54}{4x+3}=54+3x-16$;

or $(20-x)(4x+3)=72x-54$; &c.

7. $\frac{10x}{x+4}-\frac{6x-4}{2x-3}=4$; or $10-\frac{40}{x+4}-3-\frac{5}{2x-3}=4$;

$\therefore\ \frac{40}{x+4}+\frac{5}{2x-3}=3$; hence $85x-100=(3x+12)(2x-3)$.

8. $\frac{40x-63}{171}=\frac{5-x}{3+x}$; whence $10x^2+57x=261$; &c.

Ex. 61.

1. $(x-7)(x+3)=0$.
2. $(x-\frac{2}{3})(x+\frac{3}{2})=0$; or $(3x-2)(2x+3)=0$.
3. $(x-3)(x+3)(x-\frac{3}{4})(x+\frac{3}{4})=0$; or $(x^2-9)(16x^2-9)=0$.
4. $x(x-1)(x-2)(x-3)=0$; or $(x^2-x)(x^2-5x+6)=0$.
5. $x(x+\frac{1}{2})(x-1\frac{1}{2})(x+1)=0$; or $(x^2+x)(2x+1)(2x-3)=0$.
6. $x(x+1)(x-2)(x+2)(x-\frac{1}{4})=0$;
or $(x^2+x)(x^2-4)(4x-1)=0$.

Ex. 62.

1. (i) $3(3x+5y)+5(4x-3y)=203$; whence $x=7$
(ii) now becomes $147+2y^2=179$.

2. From (ii), $x^2 + 2xy + y^2 = 1$; subtracting (i) gives $2xy = -24$;
$\therefore x^2 - 2xy + y^2 = 49$; whence $x - y = \pm 7$; &c.

Otherwise: put $x = \frac{1}{2} + v$, $\therefore y = \frac{1}{2} - v$; $\therefore x^2 + y^2 = \frac{1}{2} + 2v^2 = 25$; $\therefore 2v^2 = 24\frac{1}{2}$; whence $v = 3\frac{1}{2}$.

3. From (i): $9x^2 + 9y^2 = 225$; from (ii) $9x^2 = (24 - 4y)^2$;
$\therefore (24 - 4y)^2 + 9y^2 = 225$; whence y can be found.

4. From (i): $x^2 - 2xy + y^2 = 30\frac{1}{4}$;
from (ii): $\quad 4xy \quad = 80$;
$x^2 + 2xy + y^2 = 110\frac{1}{4}$; $\therefore x + y = 10\frac{1}{2}$; &c.

5. (i): $x + y = \frac{66}{x}$; by which divide (ii), then $x - y = \frac{x}{6}$;

$\therefore 2x = \frac{66}{x} + \frac{x}{6}$; whence $x = \pm 6$.

6. Assume $x = z + 1$, $\therefore y = z - 1$. Accordingly, $x^2 - y^2 = 4z$, and $xy = z^2 - 1$;
hence (ii) becomes $15 \times 4z = 16(z^2 - 1)$, or $4z^2 - 15z = 4$;
whence $z = 4$, or $-\frac{1}{4}$.

7. From (ii): $y = x - 2$; hence, by (i), $\frac{x^2}{(x-2)^2} + 4 \cdot \frac{x}{x-2} = 8\frac{5}{9}$

$\therefore \frac{x}{x-2} + 2 = \pm 1\frac{1}{3}$; or $\frac{x}{x-2} = \frac{5}{3}$, or $-\frac{1}{3}$; $\therefore x = 5$, or $1\frac{7}{10}$.

8. From (i): $14x - 9y = 6$; whence $y = \frac{2}{9}(7x - 3)$,
and $y^2 = \frac{4}{81}(7x - 3)^2$.
From (ii): $4x^2 - 3y^2 = -12$; whence $y^2 = \frac{4}{3}(x^2 + 3)$;
$\therefore (7x - 3)^2 = 27(x^2 + 3)$, or $11x^2 - 21x = 36$.

9. (ii) + (i): $x^2 - xy + y^2 = 12$; from (i): $x^2 + 2xy + y^2 = 36$
$\therefore 3xy = 24$, and $xy = 8$; subtract this from (i), then $x - y = \pm 2$.

Otherwise: $(x+y)^3=x^3+y^3+3xy(x+y)=216$;
by substitution: $72+3xy\times 6=216$; &c.

Or thus: put $x=3+z$, $\therefore\ y=3-z$;
then $x^3+y^3=54+18z^2=72$; whence $z=\pm 1$.

10. (ii) gives $2x=\frac{4}{3}y$; hence (i) becomes $2y^2+\frac{4}{3}y+y=485$; &c.

11. (ii)$\div$(i): $x^2+xy+y^2=19$; from (i), $x^2-2xy+y^2=1$; &c.

Otherwise: $(x-y)^3=x^3-y^3-3xy(x-y)$; substitute now as in the 9th example.

Or thus: put $x=\frac{1}{2}(z+1)$, $\therefore\ y=\frac{1}{2}(z-1)$;
hence $x^3-y^3=\frac{1}{8}(6z^2+2)=19$; &c.

12. (i): $x^3+y^3=189$; (ii) $x+y=180\div xy$;
(i)$\div$(ii): $x^2-xy+y^2=21xy\div 20$.

From (ii): $x^2+2xy+y^2=32400\div x^2y^2$;

$$\therefore\ 3xy=\frac{32400}{x^2y^2}-\frac{21xy}{20};\ \text{whence } x^3y^3=8000;$$

$$\therefore\ xy=20,\ \text{and } y=\frac{20}{x};\ \text{hence (ii) becomes } x+\frac{20}{x}=9.$$

13. (i), $y=a-x$; hence (ii), $x^2+a^2-2ax+x^2=b^2$;
or $4x^2-4ax=2b^2-2a^2$; $\therefore\ 4x^2-4ax+a^2=2b^2-a^2$;
whence $2x-a=\pm\sqrt{2b^2-a^2}$.

14. (i): $4xy=4a^2$; (ii) $x^2-2xy+y^2=b^2$;
$\therefore\ x^2+2xy+y^2=4a^2+b^2$, and $x+y=\pm\sqrt{4a^2+b^2}$; &c.

15. Assume $x=v^3$, and $y=z^3$; then (i) $v+z=3$;
(ii) $v^3+z^3=9$; which may be solved as no. 9.

16. (i): $x(x+y)=a^2$; (ii) $y(x+y)=b^2$; $\therefore\ \frac{x}{y}=\frac{a^2}{b^2}$;

hence $\frac{x+y}{x}=\frac{a^2+b^2}{a^2}$; and from (i), $x^2\times\frac{a^2+b^2}{a^2}=a^2$;

or $x^2=\frac{a^4}{a^2+b^2}$.

Ex. 63.

1. Let x and $3x$ denote the nos.; then $(3x)^2-x^2=128$.
2. Let $5x$ and $4x$ denote the nos.; then $(5x)^2-(4x)^2=81$.
3. Let $8x$=the greater; $\therefore$ $3x$=the difference, and $5x$=the less; $(8x)^2+(5x)^2=356$.
4. Let the length$=x+8$ yards, $\therefore$ breadth$=x-8$; then $(x+8)(x-8)=960$.
5. Let $x+5$ and $x-5$ yards denote the lengths; $(x+5)^2+(x-5)^2=850$; or $x^2=400$.
6. Suppose $9x$ yds. at $4x$ shill. per yd.; $4x\times 9x=324$, or $x^2=9$.
7. Let $12x$ denote the no.; then $4x+3x+2=(2x)^2$.
8. Let x and $x-5$ denote the nos.; then $x(2x-5)=228$.
9. Let $x+6$ and $x-6$ feet denote the sides: $(x+6)^2+(x-6)^2=2120$.
10. Suppose x men in front, $x+5$ in depth; $\therefore$ whole no. of men, $x(x+5)=5(x+845)$.
11. If x denote the no., then $(x+3)(x+5)=(2x)^2-1$.
12. If the lengths be $x+3$ and $x-3$ yards, the no. of shill. in the whole payment will be $(x+3)^2+(x-3)^2=356$.
13. Let x and $x+5$ be the nos. of sheep. Half the total cost, 150 shill.,$=x$ sheep at $x+5$ shill.; $\therefore$ $x^2+5x=150$.
14. Let $7+x$ and $7-x$ denote the nos.; $\therefore$ $\dfrac{7-x}{7+x}=\dfrac{9}{16}\cdot\dfrac{7+x}{7-x}$;

    hence $\left(\dfrac{7-x}{7+x}\right)^2=\frac{9}{16}$; $\therefore$ $\dfrac{4(7-x)}{7+x}=\pm 3$.

    Thus we have $28-4x=21+3x$, or $x=1$;

    also $28-4x=-21-3x$, or $x=49$.

15. Let x=the greater, y the less; then $\dfrac{x+y}{x-y}=\dfrac{x}{y}$;

    hence $x^2-xy=xy+y^2$, or $x^2-2xy+y^2=2y^2$

    $\therefore$ $x=y\pm\sqrt{2}$, and $\dfrac{x}{y}=1+\sqrt{2}$.

16. If $x+3$ be the hypot., the sides are x and $x-3$; and (by Eucl. I. 47) we have $(x+3)^2=x^2+(x-3)^2$; or $x^2-6x=6x$; $\therefore\ x=12$.

17. Suppose x oxen bought at $\frac{240}{x}$ pounds a head;

$$8x-3\left(\frac{240}{x}+8\right)=59;\ \text{or } 8x^2-83x=720.$$

18. Suppose x yds. bought at $\frac{2940}{x}$ shill. per yd.

$$(x-12)\left(\frac{2940}{x}+5\right)=2405;\ \text{or } 1+\frac{588}{x}=\frac{481}{x-12};$$

$$\therefore\ x-12+\frac{588x-7056}{x}=481;\ \text{or } x^2+95x-7056.$$

19. Let $x-6$ inches=width of frame; $2(x-6)+18$, or $2x+6$, is the length of the rectangle composed of glass and frame, and $2(x-6)+12$, or $2x$, is the width of ditto; hence area of ditto $2x(2x+6)=18\times12\times2$;

$$\therefore\ x^2+3x=108.$$

20. Let x=no. by A, $x-5$ by B; $\therefore\ \frac{100}{x-5}-\frac{100}{x}=1$;

hence $x(x-5)=500$.

21. Let x yds.=circumf. of fore wheel; $\frac{120}{x}$=no. of its revolutions; $\frac{120}{x}-6$, or $\frac{120-6x}{x}$=no. of revolutions by hind wheel, the circumf. of which, therefore, is $\frac{120x}{120-6x}$, or $\frac{20x}{20-x}$ yards.

$$\frac{120}{x+1}-120\div\left(\frac{20x}{20-x}+1\right)=4;$$

$$\text{or } \frac{1}{x+1}-\frac{20-x}{19x+20}=\tfrac{1}{30};\ \therefore\ \frac{x^2}{19x^2+39x+20}=\tfrac{1}{60};$$

whence $11x^2-39x=20$.

22. Suppose it cost x pounds:

$$x-\frac{x}{100}\text{ of }x=24;\text{ or }100x-x^2=2400,$$

23. Let x pounds=prime cost per yard. The profit per yard, in pounds, is $\frac{17\frac{1}{2}}{20}\div 80$, or $\frac{69}{6400}$;

$$\therefore \frac{8x}{100}\text{ of }\frac{x}{3}+\frac{7\frac{1}{2}x}{100}\text{ of }\frac{2x}{3}=\frac{69}{6400};$$

$$\text{or }\frac{8x^2+15x^2}{300}=\frac{69}{6400},\text{ or }6400x^2=900.$$

24. Suppose A embarked x pounds, B $500-x$.

2 years' profit on x pounds$=297-x$;

$\therefore$ 2 years' profit on 1 pound$=\frac{1}{x}(297-x)$;

2 years' profit on $500-x$ pounds$=\frac{500-x}{x}(297-x)$

$\therefore$ B's 8 years' profit on $500-x$ pounds is

$$\frac{500-x}{x}(1188-4x)=297-(500-x);$$

$$\therefore \frac{1188}{x}-4=\frac{297}{500-x}-1;\text{ or }\frac{4}{x}-\frac{1}{500-x}=\frac{1}{99};$$

whence $x^2-995x=-198000$.

25. Let x, y, z, denote the nos.

(i) $x(y+z)=26$; (ii) $y(x+z)=50$; (iii) $z(x+y)=56$.

(iii)$-$(ii): $xz-xy=6$; but, (i), $xz+xy=26$;

$\therefore xz=16$, and $xy=10$; hence $z=\frac{16}{x}$, and $y=\frac{10}{x}$;

$\therefore$ (ii) becomes $\frac{10}{x}\left(x+\frac{16}{x}\right)=50$, or $10(x^2+16)=50x^2$.

26. Suppose the price of the 3 per Cents$=x$, $\therefore$ that of the $3\frac{1}{2}$ per Cents$=x+12\frac{3}{4}$; also, let y pounds$=$the income of each.

A invested $\frac{xy}{3}$ pounds, and B $\frac{(x+12\frac{3}{4})y}{3\frac{1}{2}}=\frac{xy}{3}+20$;

hence $153y-2xy=840$; $\therefore$ $y=\frac{840}{153-2x}$.

Again: $\left(\frac{xy}{3}+1855\right)\times\frac{3}{x+2\frac{1}{8}}=y+72$;

hence $576x+17y=43296$; $\therefore$ $y=\frac{43296-576x}{17}$;

$\therefore$ $\frac{70}{153-2x}=\frac{3608-48x}{17}$; whence $x=72\frac{1}{4}$; &c.

27. Suppose A invests x pounds when the price of the 3 per Cents is y; his stock will be $\frac{100x}{y}$, and his income $\frac{3x}{y}$.

B invests $\frac{100x}{y}$ pounds, at $y+3$; $\therefore$ his income$=\frac{300x}{y(y+3)}$.

$x+\frac{100x}{y}=15050$; or $\frac{y+100}{y}=\frac{15050}{x}$; (i)

$\frac{300x}{y(y+3)}-\frac{3x}{y}=87\frac{1}{2}$; or $\frac{600}{y(y+3)}-\frac{6}{y}=\frac{175}{x}$; (ii)

(i)$\div$(ii): $\frac{(y+100)(y+3)}{582-6y}=86$; $\therefore$ $y^2+619y=49752$;

whence $y=72$; and from (i) $x=\frac{15050\times72}{172}=$£6300.

Ex. 64.

1. $x^4-5x^2+\frac{25}{4}=\frac{9}{4}$; $\therefore$ $x^2=\frac{5}{2}\pm\frac{3}{2}$.

2. $\frac{1}{x^2}-\frac{2}{x}+1=9$; $\therefore$ $\frac{1}{x}=1\pm3$.

3. $(x^2-2)^2=\frac{1}{4}(x^2-2+14)$;
$\therefore (x^2-2)^2-\frac{1}{4}(x^2-2)+\frac{1}{64}=\frac{225}{64}$;
whence $x^2-2-\frac{1}{8}=\pm\frac{15}{8}$.

4. Mult. by $x^{\frac{1}{2}}$: then $2x+2=5\sqrt{x}$, or $x-\frac{5}{2}\sqrt{x}=-1$.

5. Mult. by $x^{\frac{3}{4}}$: then $1+27x^{\frac{6}{4}}=28x^{\frac{3}{4}}$; or $x^{\frac{6}{4}}-\frac{28}{27}x^{\frac{3}{4}}=-1$;
whence $x^{\frac{3}{4}}=\frac{14}{27}\pm\frac{13}{27}$; $\therefore x^{\frac{1}{4}}=1$ or $\frac{1}{3}$.

6. $(x^2-9)^2=3+11(x^2-9+7)$; $\therefore (x^2-9)^2-11(x^2-9)=80$.
Assume $x^2-9=z$: then $z^2-11z=80$;
whence $z=16$, or -5.

7. $2x^4-10x^2+14=x^4+5$; or $x^4-10x^2=-9$.

8. Assume $x^{\frac{1}{3}}=z$: then $4z^2(z^2-2)=9z^2-4$;
whence $z=\pm 2$, or $\pm\frac{1}{2}$.

9. $x^{-1}+2x^{-\frac{2}{3}}+5x^{-\frac{1}{3}}+10=x^{-1}+8$; $\therefore x^{-\frac{2}{3}}+\frac{5}{2}x^{-\frac{1}{3}}=-1$;
whence $x^{-\frac{1}{3}}=-\frac{1}{2}$, or -2; $\therefore x^{\frac{1}{3}}=-2$, or $-\frac{1}{2}$.

10. $-x^4+x^3+4x^2+3x+1=x^3-x^2+3x+5$;
or $x^4-5x^2=-4$.

11. Assume $\frac{1}{x^n}=z$: then $\frac{z}{1+z}+\frac{1-z}{z}=\frac{1}{6}$; or $\frac{1}{z^2+z}=\frac{1}{6}$;
$\therefore z^2+z=6$; whence $z=2$, or -3; $\therefore x^n=\frac{1}{2}$, or $-\frac{1}{3}$.

12. $x^4-\frac{5}{4}x^2=-\frac{1}{4}$.

13. Assume $(x+7)^{\frac{1}{4}}=z$: then $z+\frac{3}{4}z^2=5$.

14. $7x^2-5x+8\sqrt{(7x^2-5x+1)}=-8$;
$\therefore (7x^2-5x+1)+8\sqrt{(7x^2-5x+1)}=-7$.
Assume $\sqrt{(7x^2-5x+1)}=z$: then $z^2+8z=-7$;
whence $z=-1$, or -7.
$\therefore 7x^2-5x+1=1$; whence $x=0$, or $\frac{5}{7}$;
also, $7x^2-5x+1=49$; whence $x=3$, or $-2\frac{2}{7}$.

15. $(11x-2x^2+2)-14\sqrt{(11x-2x^2+2)}=-40$.
Assume $\sqrt{(11x-2x^2+2)}=z$: $\therefore\ z^2-14z=-40$;
whence $z=10$, or 4; $\therefore\ 11x-2x^2+2=16$;
also $11x-2x^2+2=100$.

16. $2x^2-7x+6+2\sqrt{(2x^2-7x+6)}=0$;
or, by assumption, $z^2+2z=0$; whence $z=0$, or -2.
Thus we have $2x^2-7x+6=0$, and also $=4$.

17. $9x-3x^2-11+4\sqrt{(x^2-3x+5)}=0$;
$\therefore\ 3(x^2-3x+5)-4\sqrt{(x^2-3x+5)}=4$;
or, by assumption, $z^2-\frac{4}{3}z=\frac{4}{3}$; whence $z=2$, or $-\frac{2}{3}$.
$\therefore\ x^2-3x+5=4$, or $=\frac{1}{3}$.

18. $\frac{1}{a}(x^2-ax)+b=\sqrt{(x^2-ax+b^2)}$;
$\therefore\ (x^2-ax+b^2)-a\sqrt{(x^2-ax+b^2)}=b^2-ab$;
whence $\sqrt{(x^2-ax+b^2)}=a-b$, or $=b$;
$\therefore\ x^2-ax+b^2=a^2-2ab+b^2$, or $=b^2$;
whence $x(x-a)=0$; $\therefore\ x=0$, or a;
also $x^2-ax=a^2-2ab$; $\therefore\ x=\frac{1}{2}\{a\pm\sqrt{a(5a-8b)}\}$.

19. $2x^2-2x+10\sqrt{(2x^2-5x+6)}=3x+33$;
$(2x^2-5x+6)+10\sqrt{(2x^2-5x+6)}=39$;
whence $\sqrt{(2x^2-5x+6)}=3$, or -13; &c.

20. $4x^2-9x-\sqrt{(4x^2-9x+11)}=-5$;
$(4x^2-9x+11)-\sqrt{(4x^2-9x+11)}=6$;
whence $\sqrt{(4x^2-9x+11)}=3$, or -2; &c.

21. $(x+4)(x-4)+\sqrt{(x+4)}\cdot\frac{x-4}{\sqrt{(x-4)}}=12$;
or $(x^2-16)+\sqrt{(x^2-16)}=12$;
whence $\sqrt{(x^2-16)}=3$, or -4; &c.

22. $\left(\frac{1+x}{1-x}\right)^{\frac{2}{5}}+\frac{3}{16}=\frac{(1-x^2)^{\frac{1}{5}}}{(1-x)^{\frac{2}{5}}}=\left(\frac{1+x}{1-x}\right)^{\frac{1}{5}}$.

Put $\left(\frac{1+x}{1-x}\right)^{\frac{1}{5}}=z$: $\therefore\ z^2-z=-\frac{3}{16}$; whence $z=\frac{3}{4}$, or $\frac{1}{4}$;

$\therefore\ \frac{1+x}{1-x}=\frac{243}{1024}$, or $\frac{1}{1024}$; hence $\frac{2x}{2}=-\frac{781}{1267}$, or $-\frac{1023}{1025}$.

23. Squaring: $x^4-14x^2+49=x^2-42x+89$;
or $x^4-6x^2-8x^2+9+40=x^2-42x+89$;
$\therefore\ x^4-6x^2+9=9x^2-42x+49$;
that is, $(x^2-3)^2=(3x-7)^2$; $\therefore\ x^2-3=\pm(3x-7)$;
from $x^2-3=-3x+7$ we have $x=2$, or -5;
from $x^2-3=3x-7$ we have $x=\frac{1}{2}(3\pm\sqrt{-7})$.

24. Mult. by $2x^2$: then $x^6+126x=441+98x^4-2392x^2$;
$\therefore\ x^6-98x^4+2401x^2=9x^2-126x+441$;
that is, $x^2(x^2-49)^2=(3x-21)^2$;
hence $x(x^2-49)=\pm3(x-7)$.
Each side is divisible by $x-7$; $\therefore\ x-7=0$, or $x=7$;
also, the division gives $x(x+7)=\pm3$; &c.

Ex. 65.

1. $x^2+\frac{1}{x^2}+x+\frac{1}{x}=4$; $\therefore\ \left(x^2+2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=6$;
whence $x+\frac{1}{x}=2$, or -3.
First, $x^2-2x=-1$; $\therefore\ x-1=\pm0$; hence $=1+0$ or $1-0$:
two equal roots.
Secondly, $x^2+3x=-1$; whence $x=\frac{1}{2}(-3\pm\sqrt{5})$.

2. $3x^2-15x-4\sqrt{(x^2-5x+5)}=-16$;
$3(x^2-5x+5)-4\sqrt{(x^2-5x+5)}=-1$.
Put $\sqrt{(x^2-5x+5)}=z$, which will be found $=1$, or $\frac{1}{3}$; &c.

3. $x^2+\frac{1}{x^2}+x-\frac{1}{x}=2$; $\therefore \left(x^2-2+\frac{1}{x^2}\right)+\left(x-\frac{1}{x}\right)=0$;

whence $x-\frac{1}{x}=0$, or -1; &c.

4. Since 1 is obviously one of the roots, divide by $x-1$: then

$x^4+x^3+x^2+x+1=0$; $\therefore x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=0$;

hence $\left(x^2+2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=1$;

$\therefore x+\frac{1}{x}=\frac{1}{2}(-1\pm\sqrt{5})$; or $x^2-\frac{1}{2}(1\pm\sqrt{5})x=-1$;

$x^2-\frac{1}{2}(-1\pm\sqrt{5})x+\frac{1}{16}(\mp 2\sqrt{5}+6)=\frac{1}{16}(\mp 2\sqrt{5}-10)$;
&c.

5. $x^2+1=p(x+1)$. Since each side is divisible by $x+1$, we have $x+1=0$, or $x=-1$.

The division gives $x^2-x+1=p$;
$\therefore x^2-x+\frac{1}{4}=\frac{1}{4}(4p-3)$; &c.

6. $(2x^3-2x^2)+(x^2-1)=0$; or
$2x^2(x-1)+(x+1)(x-1)=0$; $\therefore x-1=0$, or $x=1$;
also $2x^2+x+1=0$; &c.

7. [*See* (147), Ex. 4.] Cubing gives

$$(1+2x)+(1-2x)+3\sqrt[3]{c(1-4x^2)}=c;$$

hence $2+3\sqrt[3]{c(1-4x^2)}=c$; or $3\sqrt[3]{c(1-4x^2)}=c-2$;

$\therefore 27c(1-4x^2)=(c-2)^3$; and $\therefore 4x^2=1-\frac{(c-2)^3}{27c}$.

8. $(x^3+1)+px(x+1)=0$. This being divisible by $x+1$, we have $x+1=0$, or $x=-1$.

The division gives $x^2-x+1+px=0$; or
$x^2-(1-p)x=-1$.

9. $(n^2+1)x^2-4nax=-(n^2+1)(a^2-c^2)$;

$x^2-\frac{4na}{n^2+1}x=c^2-a^2$;

$x^2-\frac{4na}{n^2+1}x+\frac{4n^2a^2}{(n^2+1)^2}$

$=\frac{c^2(n^2+1)^2-a^2(n^4+2n^2+1-4n^2)}{(n^2+1)^2}$; &c.

10. $36x-3^46x^3-19x^2-36x+36=0$: a recurring equation.

Divide by x^2: then $36x^2-36x-19-\frac{36}{x}+\frac{36}{x^2}=0$;

$\therefore \left(36x^2+\frac{36}{x^2}\right)-\left(36x+\frac{36}{x}\right)=19$;

hence $\left(36x^2+72+\frac{36}{x^2}\right)-6\left(6x+\frac{6}{x}\right)=91$.

Put $6x+\frac{6}{x}=z$, which will be found $=13$, or -7;

hence we have $6x^2-13x=-6$; also $6x^2+7x=-6$.

11. $x^4-1=3x(x^2-1)$; $\therefore x^2-1=0$; also $x^2+1=3x$.

12. $x^3-x=2(x+1)$; $\therefore x+1=0$; also $x(x-1)=2$.

13. Assume $x=z^2$: then $z^2-1=2\left(1+\frac{1}{z}\right)$;

$z(z^2-1)=2(z+1)$. Divide by $z+1$:
$z(z-1)=2$; whence $z=2$, or -1;
also, $z+1=0$ gives $z=-1$.

14. Divide by $\sqrt[3]{(1-x)^2}$, then

$\sqrt[3]{\left(\frac{1+x}{1-x}\right)^2}-1=\sqrt[3]{\frac{1-x^2}{(1-x^2)}}=\sqrt[3]{\frac{1+x}{1-x}}$, for which write z:

then $z^3-1=z$; whence $z=\frac{1}{2}\sqrt{(1\pm\sqrt{5})}$;

$\therefore z^3=\frac{1}{8}(16\pm8\sqrt{5})=2\pm\sqrt{5}$;

hence $\frac{1+x}{1-x}=\frac{2\pm\sqrt{5}}{1}$; $\therefore \frac{2x}{2}=\frac{1\pm\sqrt{5}}{3\pm\sqrt{5}}$.

Mult. numerator and denominator by $3\mp\sqrt{5}$

then $\frac{-2\pm2\sqrt{5}}{9-5}=\frac{1}{2}(-1\pm\sqrt{5})=x$.

15. $ax-3nx+n+2\sqrt{nx}\sqrt{(n+ax)}=0$;
$(n+ax)+2\sqrt{nx}\sqrt{(n+ax)}=3nx$;
$(n+ax)+2\sqrt{nx}\sqrt{(n+ax)}+nx=4nx$;
$\therefore\ \sqrt{n+ax}+\sqrt{nx}=\pm 2\sqrt{nx}$;
hence $\sqrt{n+ax}=\sqrt{nx}$, and $=-3\sqrt{nx}$;
thus we have $n+ax=nx$, and also $=9nx$.

16. $x(\sqrt{x}-\sqrt{b})-a(\sqrt{x}+\sqrt{b})=b(\sqrt{x}-\sqrt{b})$;
$\therefore\ a(\sqrt{x}+\sqrt{b})=(x-b)(\sqrt{x}-\sqrt{b})$;
this is divisible by $\sqrt{x}+\sqrt{b}$; $\therefore\ \sqrt{x}+\sqrt{b}=0$, or $x=b$;
the division gives $a=(\sqrt{x}-\sqrt{b})^2=x-2\sqrt{bx}+b$;
$\therefore\ x-2\sqrt{b}\sqrt{x}=a-b$; hence $\sqrt{x}-\sqrt{b}=\pm\sqrt{a}$.

17. $x^3-1=2px(x-1)$. Here the factor $x-1=0$, or $x=1$;
also $x^2+x+1=2px$; or $x^2+(1-2p)x=-1$; &c.

18. $x^4+1-3(x^3-x)=\frac{4}{9}x^2$; $\therefore\ x^2+\frac{1}{x^2}-3\left(x-\frac{1}{x}\right)=\frac{4}{9}$;
$\left(x^2-2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)=-1\frac{4}{9}$;
whence $x+\frac{1}{x}=2\frac{1}{3}$, or $\frac{2}{3}$; &c.

19. $x^2-\frac{4}{9x}=1\frac{3}{9}$; or $9x^3-4=13x$; $\therefore\ 9x^3-9x=4x+4$.
Here we have a factor, $x+1=0$; $\therefore\ x=-1$;
also $x(x-1)=\frac{4}{9}$; &c.

20. $x^{12}-1=0=(x^6+1)(x^6-1)=(x^2+1)(x^4-x^2+1)$
$=(x^2-1)(x^4+x^2+1)$.
Put each factor$=0$, and derive x from each equation.

21. $(x^4+1)-\frac{1}{2}(x^3-x)=\frac{7}{2}x^2$; $\therefore\ \left(x^2+\frac{1}{x^2}\right)-\frac{1}{2}\left(x-\frac{1}{x}\right)=\frac{7}{2}$;
$\left(x^2-2+\frac{1}{x^2}\right)-\frac{1}{2}\left(x-\frac{1}{x}\right)=\frac{3}{2}$; or $z^2-\frac{1}{2}z=\frac{3}{2}$; &c.

22. Assuming, for convenience, $y^3=x$, $m^3=a$, $n^3=b$, we have
$y^3+m^3+3mny=n^3$; or $y^3+m^3=n^3-3mny$;
that is, $(y+m)^3-3my(y+m)=n^3-3mny$;
hence $(y+m)^3-n^3=3my\{(y+m)-n\}$.

This is divisible by $(y+m)-n$; so that we have
$y+m-n=0$; or $y=n-m$ is one value of y.

The division gives $(y+m)^2+n(y+m)+n^2=3my$,
or $y^2-(m-n)y=-m^2-mn-n^2$;
$\therefore\ y^2-(m-n)y+\frac{1}{4}(m-n)^2=-\frac{3}{4}(m+n)^2$;
whence $y=\frac{1}{2}\{m-n\pm(m+n)\sqrt{-3}\}$;

$\therefore\ y^3=x=\frac{1}{8}\{a^{\frac{1}{3}}-b^{\frac{1}{3}}\pm(a^{\frac{1}{3}}+b^{\frac{1}{3}})\sqrt{-3}\}^3$.

Also, we had $y=n-m$; $\therefore\ y^3=(n-m)^3$;
that is, $x=\left(b^{\frac{1}{3}}-a^{\frac{1}{3}}\right)^3$.

23. $-x^4+2x^2+9+\sqrt{(x^2+9)}=0$;
or $(x^2+9)+\sqrt{(x^2+9)}=x^4-x^2$;
whence $\sqrt{(x^2+9)}+\frac{1}{2}=\pm(x^2-\frac{1}{2})$;
or $\sqrt{(x^2+9)}=x^2-1$, and also $=-x^2$;
thus we have $x^2+9=x^4-2x^2+1$; also $x^2+9=x^4$.

24. $4x^2+12x\sqrt{(1+x)}+9(1+x)=36(1+x)$;
$\therefore\ 2x+3\sqrt{(1+x)}=\pm6\sqrt{(1+x)}$.
Thus we have $4x^2=9(1+x)$, and also $=81(1+x)$.

25. $(x^2+2x+1)-8(x+1)\sqrt{x}=-16x$;
or $(x+1)^2-8(x+1)\sqrt{x}+16x=0$;
hence $x+1-4\sqrt{x}=0$; $\therefore\ x^2+2x+1=16x$; &c.

26. $x^4-4x^2+1=11(x^2+x\sqrt{2}-1)$.

This is divisible by $x^2+x\sqrt{2}-1$;
therefore $x^2+x\sqrt{2}-1=0$;
whence $x=\frac{1}{2}(\pm6-\sqrt{2})$.

The division gives $x^2-x\sqrt{2}-1=11$;
$\therefore\ x^2-x\sqrt{2}+\frac{1}{2}=12\frac{1}{2}$;
whence $x-\frac{1}{2}\sqrt{2}=\pm\frac{5}{2}\sqrt{2}$.

27. By (88): $\frac{\sqrt{x}}{\sqrt{(x-a)}}=\frac{n^2a+(x-a)}{n^2a-(x-a)}$;

$\therefore \frac{x}{x-a}=\frac{n^4a^2+2n^2a(x-a)+(x-a)^2}{n^4a^2-2n^2a(x-a)+(x-a)^2}$;

hence $\frac{x}{a}=\frac{\{n^2a+(x-a)\}^2}{4n^2a(x-a)}$;

$\therefore 4n^2x^2-4n^2ax=n^4a^2+2n^2ax-2n^2a^2+x^2-2ax+a^2$;

$(4n^2-1)x^2-2(3n^2-1)ax=(n^2-1)^2a^2$;

$\therefore x^2-\frac{2a(3n^2-1)}{4n^2-1}x=\frac{a^2(n^2-1)^2}{4n^2-1}$;

hence $x=\frac{a(3n^2-1)}{4n^2-1}\pm\frac{2an^3}{4n^2-1}=\frac{a(\pm2n^3+3n^2-1)}{(2n+1)(2n-1)}$

$=\frac{a(1+n)^2}{1+2n}$, and also $=\frac{a(1-n)^2}{1-2n}$.

28. $a(1-4x+6x^2-4x^3+x^4)=1+x^4$;

or $(a-1)(x^4+1)-4a(x^3+x)=-6ax^2$;

$\therefore (a-1)\left(x^2+\frac{1}{x^2}\right)-4a\left(x+\frac{1}{x}\right)=-6a$;

$\left(x^2+\frac{1}{x^2}\right)-\frac{4a}{a-1}\left(x+\frac{1}{x}\right)=-\frac{6a}{a-1}$;

$\therefore\left(x^2+2+\frac{1}{x^2}\right)-\frac{4a}{a-1}\left(x+\frac{1}{x}\right)=\frac{-2-4a}{a-1}$;

hence $x+\frac{1}{x}=\frac{2a\pm\sqrt{2(a+1)}}{a-1}$, $=p$, suppose.

$\therefore x^2-px=-1$; &c.

29. By (88): $\frac{a+x}{\sqrt{(a^2-x^2)}}=\frac{c+x}{c-x}$; or $\frac{\sqrt{(a+x)}}{\sqrt{(a-x)}}=\frac{c+x}{c-x}$;

$\therefore \frac{a+x}{a-x}=\frac{c^2+2cx+x^2}{c^2-2cx+x^2}$; $\therefore$ by 88: $\frac{x}{a}=\frac{2cx}{c^2+x^2}$.

This divides by x, and $\therefore x=0$ is one value.
The division gives $c^2+x^2=2ac$; &c.

30. By (88): $\frac{a-\sqrt{a^2-ax}}{a}=\left(\frac{1-c}{1+c}\right)^2$;

again, by (88): $\frac{\sqrt{a^2-ax}}{a}=\frac{4c}{(1+c)^2}$; $\therefore \frac{a^2-ax}{a^2}=\frac{16c^2}{(1+c)^4}$;

or $1-\frac{x}{a}=\frac{16c^2}{(1+c)^4}$; $\therefore x=a\left(1-\frac{16c^2}{(1+c)^4}\right)$.

31. By (88): $\frac{ax+1}{\sqrt{(a^2x^2-1)}}=\frac{b^2x+2}{b^2x-2}$;

$\therefore \frac{ax+1}{ax-1}=\frac{b^4x^2+4b^2x+4}{b^4x^2-4b^2x+4}$; and, by (88), $\frac{ax}{1}=\frac{b^4x^2+4}{4b^2x}$;

$\therefore b^2(4a-b^2)x^2=^4$; whence $x=\pm\frac{2}{b}\sqrt{(4a-b^2)}$.

32. From (i): $x^2+y^2=25-2xy$:
also, $x^4+y^4=625-4xy(x^2+y^2)-6x^2y^2$;
$\therefore 97=625-4xy(25-2xy)-6x^2y^2$; whence $xy=6$, or 44;
hence $4xy=36$, or 176, subtracted from the square of (1), gives $x^2-2xy+y^2=1$, or -151;
$\therefore x-y=\pm1$, or $\pm\sqrt{-151}$; &c.

Or thus: put $x=\frac{1}{2}(5+z)$, $y=\frac{1}{2}(5-z)$;
$\therefore x^4+y^4=\frac{1}{8}(625+150z^2+z^4)=97$;
whence $z=\pm1$, or $\pm\sqrt{-151}$.

33. From (i): $x^2+y^2=4+2xy$;
also, $x^4+y^4=16+4xy(x^2+y^2)-6x^2y^2=272$;
$\therefore 256=4xy(4+2xy)-6x^2y^2$; whence xy.

Or thus: put $x=z+1$, $\therefore y=z-1$; &c., as in the 32nd example.

34 Divide (i) by (ii), and subtract from the 4th power of (ii);

then $5xy(x^2+xy+y^2)=b^4-\frac{a^5}{b}=5xy(b^2-xy)$; whence xy.

Or thus: put $x=\frac{1}{2}(b+z)$, $\therefore y=\frac{1}{2}(b-z)$;
$\therefore x^5+y^5=\frac{1}{16}(b^5+10b^3z^2+5bz^4)=a^5$; &c.

35. (i): $\frac{2-x^2-y^2}{(1-y)(1-x)}=\frac{9}{13}$; (ii) $\frac{2-x^2-y^2}{(1+y)(1-x)}=\frac{4}{13}$;

(i)÷(ii): $\frac{1+y}{1-y}=\frac{9}{4}$; $\therefore\ y=\frac{5}{13}$.

From (i), by substitution: $\frac{13(1+x)}{8}+\frac{18}{13(1-x)}=\frac{9}{13}$;

$\therefore\ 169(1-x^2)+144=72(1-x)$; or $169x^2-72x=241$;

whence $x=-1$, or $2\frac{41}{169}$.

36. From (i): $x^2y^2=a^2(x^2+2xy+y^2)=b^2(x^2+y^2)$;

whence $x^2+y^2=-\frac{2a^2}{a^2-b^2}\cdot xy=\frac{x^2y^2}{b^2}$.

This divides by xy; whence we have $x=0$, $y=0$;

and the division gives $xy=-\frac{2a^2b^2}{a^2-b^2}$.

From (i): $x+y=-\frac{2ab^2}{a^2-b^2}$; (A)

from (ii): $x^2+y^2=\frac{4a^4b^4}{(a^2-b^2)^2}$;

hence $x^2-2xy+y^2=\frac{4a^2b^2}{(a^2-b^2)^2}\cdot(2a^2-b^2)$;

$\therefore\ x-y=\pm\frac{2ab}{a^2-b^2}\sqrt{2a^2-b^2}$. (B)

Now $\frac{1}{2}\{(A)\pm(B)\}$ will give x and y.

37. Squaring gives $\frac{3x}{x+y}+\frac{x+y}{3x}+2=4$.

Mult. by $\frac{3x}{x+y}=z$: then $z^2+1=2z$;

whence $z=1=\frac{3x}{x+y}$; $\therefore\ y=2x$;

from (ii): $x^2-\frac{3}{2}x=27$; whence $x=6$, or $-4\frac{1}{2}$.

38. In (i) put $y=vx$: then $3x^2+4v^2x^2=7vx^2$;
$\therefore\ 4v^2-7v=-3$; whence $v=1$, or $\frac{3}{4}$; and $y=x$, or $\frac{3}{4}x$.
(ii) becomes, first, $x^{\frac{3}{2}}-\frac{3}{8}x^3=x^{\frac{3}{2}}$; whence $x=0$, $y=0$.
(ii) becomes, secondly, $x^{\frac{3}{2}}-\frac{3}{8}$ of $\frac{9}{16}x^2=\frac{3}{4}x^{\frac{3}{2}}$; or $\frac{1}{4}x^{\frac{3}{2}}=\frac{1}{8}x^2$;
or $2x^{\frac{3}{2}}=x^2$; $\therefore\ 4x^3=x^4$; whence $x=4$, $y=3$.

39. (i) $x^{\frac{4}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})=208$;
(ii) $y^{\frac{4}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})=1053$.
(i)$\div$(ii): $x^{\frac{4}{3}}\div y^{\frac{4}{3}}=16\div 81$; $\therefore\ \sqrt[3]{x}\div\sqrt[3]{y}=2\div 3$;
and $x\div y=8\div 27$; $\therefore\ y=\frac{27}{8}x$.
From (i): $x^{\frac{4}{3}}(x^{\frac{2}{3}}+\frac{9}{4}x^{\frac{2}{3}})=\frac{13}{4}x^2=208$; $\therefore\ x^2=64$.

40. $\left(\frac{x^2}{y^2}+\frac{2x}{\sqrt{y}}+y\right)+\left(\frac{x}{y}+\sqrt{y}\right)=20$; $\therefore\ \frac{x}{y}+\sqrt{y}=4$, or -5.
First, $x+y^{\frac{3}{2}}=4y$; $\therefore$ from (ii): $y^{\frac{3}{2}}=8$; hence $y=4$.
Secondly, $x+y^{\frac{3}{2}}=-5y$; $\therefore$ from (ii): $y^{\frac{3}{2}}+9y-8=0$;
divide by $y^{\frac{1}{2}}+1$: then $y+8y^{\frac{1}{2}}-8=0$;
whence $y=40\pm16\sqrt{6}=8(5\pm2\sqrt{6})$; also $y^{\frac{1}{2}}=-1$, or $y=1$.

41. $\frac{x^2-y^2}{xy}=\frac{x+y}{x^2+y^2}$; or $\frac{x-y}{xy}=\frac{1}{x^2+y^2}$; (i)
$\frac{x^4-y^4}{x^2}=\frac{x-y}{1}$; or $\frac{(x^2+y^2)(x+y)}{x^2}=1$. (ii)
By multiplication: $\frac{x+y}{x^2}=\frac{x-y}{xy}$; or $y(x+y)=x(x-y)$;
$\therefore\ x^2-2xy+y^2=2y^2$; whence $x-y=\pm y\sqrt{2}$;
or $x=y(1\pm\sqrt{2})$; $\therefore\ x^2=y^2(3\pm\sqrt{2})$, and $x^2+y^2=2y^2(2\pm\sqrt{2})$.

From (ii): $2y^2(2\pm\sqrt{2})(2\pm\sqrt{2})y=y^2(3\pm2\sqrt{2})$;
or $y^2(12\pm8\sqrt{2})=y^2(3\pm2\sqrt{2})$;
$\therefore\ y=0$, or $\frac{1}{4}$; and $\therefore\ x=0$, or $\frac{1}{4}(1\pm\sqrt{2})$.

42. (i) $x^2+y^2=a^2-xy$; (ii) $x+y=b-\sqrt{xy}$.

(ii) squared gives $x^2+y^2=b^2-2b\sqrt{xy}-xy$;

$\therefore\ b^2-2b\sqrt{xy}=a^2$; also $\sqrt{xy}=\dfrac{b^2-a^2}{2b}$; and xy
$=\dfrac{b^4-2a^2b^2+a^4}{4b^2}$.

From (ii): $(x+y)^2=\left(\dfrac{a^2+b^2}{2b}\right)^2$; from which subtract $4xy$;

then $x^2-2xy+y^2=\dfrac{1}{4b^2}\{10a^2b^2-3(a^4+b^4\}$;

thus we have $x-y=\pm\dfrac{1}{2b}\sqrt{\{10a^2b^2-3(a^4+b^4)\}}$,

and $x+y=\dfrac{a^2+b^2}{2b}$; whence x and y.

43. (i) $\times$ (ii) gives $(xy)^{m+n}=a^{2m}b^{2n}cd$;

(i) $\div$ (ii) gives $(xy^{-1})^{m-n}=cd^{-1}$;

$\therefore\ xy=(a^{2m}b^{2n}cd)^{\frac{1}{m+n}}$, and $xy^{-1}=(cd^{-1})^{\frac{1}{m-n}}$.

By multiplication: $x^2=\left(a^{2m}b^{2n}c^{\frac{2m}{m-n}}d^{\frac{-2n}{m-n}}\right)^{\frac{1}{m+n}}$;

$$\therefore\ x=\left(a^mb^nc^{\frac{m}{m-n}}d^{\frac{-n}{m-n}}\right)^{\frac{1}{m+n}}.$$

By division: $y^2=\left(a^{2m}b^{2n}c^{\frac{-2n}{m-n}}d^{\frac{2m}{m-n}}\right)^{\frac{1}{m+n}}$;

$$\therefore\ y=\left(a^mb^nc^{\frac{-n}{m-n}}d^{\frac{m}{m-n}}\right)^{\frac{1}{m+n}}.$$

44. (i) $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}=\sqrt{x}$; (ii) $\frac{x+y}{x-y}=\sqrt{2}$.

From (i) $\frac{\sqrt{x}}{\sqrt{y}}=\frac{1+\sqrt{x}}{1-\sqrt{x}}$; $\therefore \frac{x}{y}=\frac{1+2\sqrt{x}+x}{1-2\sqrt{x}+x}$;

$\therefore \frac{x+y}{x-y}=\frac{1+x}{2\sqrt{x}}=\sqrt{2}$; or $\frac{1+x}{2\sqrt{2x}}=1$; $\therefore x-2\sqrt{2x}=-1$;

hence $x-2\sqrt{2x}+2=1$; $\therefore \sqrt{x}=\sqrt{2}\pm 1$.

From (ii) $\frac{y}{x}=\frac{\sqrt{2}-1}{\sqrt{2}+1}$, that is $\frac{y}{(\sqrt{2}\pm 1)^2}=\frac{\sqrt{2}-1}{\sqrt{2}+1}$

$=(\sqrt{2}-1)^2$;

whence $\frac{y}{\sqrt{2}+1}=\sqrt{2}-1$, or $y=1$; also $\frac{y}{(\sqrt{2}-1)^2}$

$=(\sqrt{2}-1)^2$, or $y=(\sqrt{2}-1)^4$.

45. (i) $ay-bx=\frac{y^4}{x^2}$; (ii) $ax-by=x^2$.

Put $y=vx$: then $avx-bx=v^4x^2$, and $ax-bvx=x^2$.

By division: $\frac{av-b}{a-bv}=v^4$; $\therefore bv^5-av^4+av-b=0$,

or $b(v^5-1)-av(v^3-1)=0$. This divides by $v-1$; so that $v=1$ is one value, $\therefore x=y$, and $x=a-bv$ becomes $x=a-b=y$.

The division gives $v^4+v^3+v^2+v+1-\frac{av}{b}(v^2+v+1)=0$.

hence $v^2+v+1+\frac{1}{v}+\frac{1}{v^2}=\frac{a}{b}\left(v+1+\frac{1}{v}\right)$;

$\therefore \left(v^2+2+\frac{1}{v^2}\right)+\left(v+\frac{1}{v}\right)=\frac{a}{b}\left(v+\frac{1}{v}\right)+\frac{a}{b}+1$;

or $\left(v+\frac{1}{v}\right)^2+\frac{b-a}{b}\left(\ +\frac{1}{v}\right)=\frac{b+a}{b}$;

whence $v+\frac{1}{v}=\frac{1}{2b}\left\{\ a-b\pm\sqrt{(a^2+2ab+5b^2)}\ \right\}=2m$, suppose.

$v^2+1=2mv$; whence $v=m\pm\sqrt{m^2-1}$;

and from (ii), $x=a-b(m\pm\sqrt{m^2-1})$;

then $y=vx=$&c.

46. (i) $(x^2-xy+y^2)(x+y)+xy(x+y)=13$,
or $(x^2+y^2)(x+y)=13$;

(i) + (ii): $\frac{x+y}{x^2y^2}=\frac{1}{3\,6}$; (A)

(i) × (A): $\frac{1}{x^2y^2}(x^4+2x^3y+2x^2y^2+2xy^3+y^4)=\frac{1\,3}{3\,6}$;

or $\frac{x^2}{y^2}+\frac{2x}{y}+2+\frac{2y}{x}+\frac{y^2}{x^2}=\frac{1\,3}{3\,6}$;

$\therefore \left(\frac{x^2}{y^2}+2+\frac{y^2}{x^2}\right)+2\left(\frac{x}{y}+\frac{y}{x}\right)+1=\frac{4\,9}{3\,6}$;

hence $\frac{x}{y}+\frac{y}{x}=-\frac{1\,3}{6}$, or $x^2+y^2=-\frac{1\,3}{6}xy$.

From (ii) $\frac{1\,3}{6}x^3y^3=-468$; or $x^3y^3=-216$;

$\therefore xy=-6$; and, from (A), $x+y=1$; &c.

47. By addition: $(x+y+z)(x+y+z)=a^2+b^2+c^2$;
$\therefore x+y+z=\pm\sqrt{(a^2+b^2+c^2)}$.
From (i): $x=a^2\div\pm\sqrt{(a^2+b^2+c^2)}$; &c.

48. (i) × (ii) × (iii) gives $x^3y^3z^3=a^3b^3c^3$; $\therefore xyz=abc$;

(i) gives $x^2y^2=\frac{a^2b^2c^2}{z^2}=c^2z$; &c.

49. Put $y=vx$: $\therefore y^2=v^2x^2=xz$; $\therefore v^2x=z$.
From (i): $x(1+v+v^2)=14$; from (ii): $x^2(1+v^2+v^4)=84$;

by division: $x(1-v+v^2)=6$; $\therefore \frac{1+v+v^2}{1-v+v^2}=\frac{7}{3}$;

$\therefore v=2$, or $\frac{1}{2}$; &c.

50. From (ii): $x^2=100-20y^2+y^4$;
from (iii): $z^2=25-10y^2+y^4$.
(i) becomes $125-29y^2+2y^4=26$; whence $y=3$, or $\frac{1}{2}\sqrt{22}$; &c.

Ex. 66.

1. (i) $x=37-7y-\frac{y-2}{4}$; $\therefore \frac{y-2}{4}=t$; hence $y=4t+2$;

$x=\frac{1}{4}(150-58-4t\times 29)=23-29t$; $\therefore\ t=0$.

(ii) $x=50-9y-\frac{2y-1}{3}$; $y-\frac{2y-1}{3}=\frac{y+1}{3}=t$;

or $y=3t-1$; and $x=\frac{1}{3}(151+29-3t\times 29)=60-29t$;
$\therefore\ t=1$, or 2.

(iii) Put $x=15v$: $\therefore\ 7v+y=15$, or $y=15-7v$;
hence $v=1$, or 2.

(iv) $y=40-2x-\frac{5x-8}{12}$; then $\frac{25x-40}{12}=2x-3+\frac{x-4}{12}$;

$\therefore \frac{x-4}{12}=t$, or $x=4+12t$;

$y=\frac{1}{12}(488-116-12t\times 29)=31-29t$; $\therefore\ t=0$, or 1.

(v) $y=93-3x-\frac{3x-4}{8}$; then $\frac{9x-12}{8}=x-1+\frac{x-4}{8}$;

hence $x=8t+4$; $y=80-27t$; and $t=0$, 1, 2.

(vi) $y=58-x-\frac{6x-2}{7}$; then $\frac{5(3x-1)}{7}=2x+\frac{x-5}{7}$;

hence $x=7t+5$; $y=49-13t$; and $t=0$, 1, 2, 3.

2. (i) $y=\frac{19x-11}{14}$, leaving $\frac{5x-11}{14}$;

then $\frac{15x-33}{14}$ leaves $\frac{x-5}{14}=t$;

hence $x=14t+5$; $y=19t+6$. We must take $t=0$.

(ii) $y=\frac{17x-1}{7}$, leaving $\frac{3x-1}{7}$;

then $\frac{15x-5}{7}$ leaves $\frac{x-5}{7}=t$;

hence $x=7t+5$; $y=17t+12$. We must take $t=0$.

(iii) $y=\frac{23x-929}{9}$, leaving $\frac{5x-2}{9}$;

then $\frac{10x-4}{9}$ leaves $\frac{x-4}{9}=t$;

hence $x=9t+4$; $y=23t-93$. We must take $t=5$.

(iv) $x=\frac{23y+19}{8}$, leaving $\frac{7y+3}{8}$;

then $y-\frac{7y+3}{8}=\frac{y-3}{8}=t$;

hence $y=8t+3$; $x=23t+11$. We must take $t=0$.

(v) $x=\frac{33y+43}{23}$, leaving $\frac{10y+20}{23}$; then $\frac{y+2}{23}=t$;

hence $y=23t-2$; $x=33t-1$. We must take $t=1$.

(vi) $y=\frac{555x-73}{22}$, leaving $\frac{5x-7}{22}$;

then $\frac{45x-63}{22}$ leaves $\frac{x-19}{22}=t$;

hence $x=22t+19$; $y=555t+476$. We must take $t=0$.

3. (i) Put $y=3v$: $\therefore x+4v=13$, or $x=13-4v$; and the no. of values of v=the integral part of the quotient of $13\div4$, viz. 3.

(ii) $x=\frac{500-13y}{8}$, leaving $\frac{4-5y}{8}$;

then $\frac{20-25y}{8}$ leaves $\frac{4-y}{8}=t$;

hence $y=4-8t$; $x=56+13t$; $\therefore$ the values of t are $0, -1, -2, -3, -4$.

(iii) $x=\frac{405-13y}{7}$, leaving $\frac{6-6y}{7}$;

$\therefore \frac{1-y}{7}=t$; hence $y=1-7t$; $x=56+13t$;

and the values of t are $0, -1\ -2\ -3, -4$.

(iv) $x=\frac{125-7y}{2}$, leaving $\frac{1-y}{2}=t$; hence $y=1-2t$;

$x=59+7t$; and the no. of values of t=the integral part of $59\div 7$, besides zero, viz. 9.

(v) $x=\frac{3640-90y}{31}$; $\frac{364-9y}{31}$ leaves $\frac{23-9y}{31}$;

then $\frac{161-63y}{31}$ leaves $\frac{6-y}{31}=t$;

hence $y=6-31t$; $x=100+90t$; $\therefore\ t=0$, or -1.

(vi) $x=\frac{943-15y}{11}$, leaving $\frac{8-4y}{11}$; then $\frac{2-y}{11}=t$;

hence $y=2-11t$; $x=83+15t$;

$\therefore\ t=0$, and $83\div 15$ gives 5 values more.

4. Addition gives $y=3x-12$; $\therefore$ the least value of x is 5; $y=3$; $z=2x+y-7=6$.

5. Given, $7x+12y=50$: to find $x+y$.

$x=\frac{50-12y}{7}$, leaving $\frac{1-5y}{7}$; then $\frac{3-15y}{7}$ leaves $\frac{3-y}{7}=t$;

hence $y=3-7t$; $x=2+12t$; $\therefore t=0$, only.

6. (i) Valuing in sixpences, we have $21x+5y=42\times 12$.
Put $y=21v$: then $x+5v=24$; $\therefore\ x=24-5v$;
and $24\div 5$ gives 4 values of v.

(ii) Valuing in shillings, we have $21x+2y=21\times 12$.
Put $y=21v$: then $x+2v=12$, or $x=12-2v$;
hence, as $v=0$ is inadmissible, the no. of values of v is $(12\div 2)-1$, or 5.

7. $\frac{x}{18}+\frac{y}{12}=\frac{25}{36}$; $\therefore\ 2x+3y=25$; and $x=\frac{25-3y}{2}$, leaving

$\frac{1-y}{2}=t$; $\therefore\ y=1-2t$, and $x=11+3t$;

hence $t=0,\ -1,\ -2,\ -3$.

8. Let A give B x half-crowns, and receive from B y groats; then, valuing in pence, $30x-4y=32\times12$; or $x=\frac{2y+192}{15}$; $\therefore \frac{y+96}{15}=t$; hence $y=15t-96$; $x=2t$; and t has every positive integral value above 6; $\therefore$ take $t=7$.

9. (i) If x and y denote the quotients, the no., N, will be denoted by $3x+2=5y+3$;

hence $x=\frac{5y+1}{3}$; $\therefore 2y-\frac{5y+1}{3}$, or $\frac{y-1}{3}=t$;

hence $y=3t+1$; $x=5t+2$;

and taking $t=0$, we have $x=2$; $\therefore N=3x+2=8$.

(ii) $N=3x+1=7y+2$; hence $3x-7y=1$; $\therefore x=\frac{7y+1}{3}$,

leaving $\frac{y+1}{3}=t$; hence $y=3t-1$; $x=7t-2$.

Least value of t is 1; $\therefore x=5$; and $N=3x+1=16$.

10. $8x+11y=300$; $\therefore x=\frac{300-11y}{8}$, leaving $\frac{4-3y}{8}$; then $\frac{12-9y}{8}$ leaves $\frac{4-y}{8}=t$; hence $y=4-8t$; $x=32+11t$; $\therefore t=0, -1, -2$, and we must take $t=-1$, which gives $x=21$, $y=12$.

11. $$\begin{array}{rl} & x+\ y+z=18 \\ \text{and, valuing in sixpences:} & \underline{5x+2y+z=40} \\ \text{hence} & 4x+\ y\ =22; \end{array}$$ $\therefore y=22-4x$;

where x may have the values 2, 3, 4, 5.

12. Given $3x+2=5y+4=N$:

$x=\frac{5y+2}{3}$; then $\frac{10y+4}{3}$ leaves $\frac{y+1}{3}=t$;

hence $y=3t-1$; $x=5t-1$; and $t=1, 2, 3, 4$, &c.

$N=3x+2=15t-1$; which by taking $t=4$ gives 59.

13. Valuing in sixpences: $21x+5y=183$; $\therefore y=\frac{183-21x}{5}$,

leaving $\frac{3-x}{5}=t$; hence $x=3-5t$; $y=24+21t$;

hence $t=0$, or -1; take $t=-1$, which gives $x=8$, $y=3$.

14. $x+y+z=40$; and valuing in crowns, $20x+4y+z=160$;

$\therefore 19x+3y=120$; or $x=\frac{120-3y}{19}$; $\therefore \frac{40-y}{19}=t$;

hence $y=40-19t$; $x=3t$; $z=16t$; and $t=1$, or 2.

15. $7x+6=8y+7=9z+8$; $\therefore x+\frac{8y+1}{7}$, leaving $\frac{y+1}{7}=p$;

hence $y=7p-1$; and $N=8y+7=56p-1=9z+8$;
$\therefore 56p=9(z+1)$; hence $p=9t$; $y=63t-1$;
$\therefore N=8y+7=504t-1$. Take $t=1$.

16. x shill. for the chicken, y for a duck, z for a goose.

We have $3x+y-2z=0$, and $x+2y+3z=25$;

whence $11x+7y=50$; $\therefore y=\frac{50-11x}{7}$, leaving $\frac{1-4x}{7}$;

then $\frac{2-8x}{7}$ leaves $\frac{2-x}{7}=t$; hence $x=2-7t$; $y=4+11t$;

$z=5-5t$; and $t=0$, only.

17. $N=2u+1=3v+1=4w+1=5x+1=6y+1=7z$.

Now we have here $u=2w$, and $v=2y$; therefore we need only the last four forms:

$4w+1=5x+1$; or $w=\frac{5}{4}x$; $\therefore x=4p$; and $5x=6y$ becomes $20p=6y$; $\therefore p=\frac{3}{10}y$; hence $y=10q$.

$6y=7z-1$ now becomes $60q=7z-1$; $\therefore q=\frac{7z-1}{60}$;

then $\frac{301z-43}{60}$ leaves $\frac{z-43}{60}=t$; hence $z=43+60t$;

taking $t=0$, we have $N=7z=43\times 7=301$.

18. $N=23x+20=27y+14$; $\therefore x=\frac{27y-6}{23}$, leaving $\frac{4y-6}{23}$;

then $\frac{24y-36}{23}$ leaves $\frac{y-13}{23}=t$; hence $y=23t+13$;

$\therefore N=27y+14=23t\times 27+365$. Take $t=0$.

19. $N=19x+18=22y+17=23z+16$.

$x=\frac{22y-1}{19}$, leaving $\frac{3y-1}{19}$;

then $\frac{39y-13}{19}$ leaves $\frac{y-13}{19}=p$;

$\therefore y=19p+13$; $N=22y+17=418p+303$.

Hence $23z+16=418p+303$;

$\therefore z=\frac{418p+287}{23}$, leaving $\frac{4p+11}{23}$;

then $\frac{24p+66}{23}$ leaves $\frac{p+20}{23}=t$; hence $p=23t-20$;

$N=418p+303=9614t-8057$. Take $t=1$.

20. $N=25x+18=29y+15=31z+6$.

We have $x=\frac{29y-3}{25}$, leaving $\frac{4y-3}{25}$;

then $\frac{76y-57}{25}$ leaves $\frac{y-7}{25}=p$; $\therefore y=25p+7$,

$N=29y+15=725p+218$.

Hence $31z+6=725p+218$;

$\therefore z=\frac{725p+212}{31}$, leaving $\frac{12p+26}{31}$;

then $\frac{156p+338}{31}$ leaves $\frac{p+28}{31}=t$; hence $p=31t-28$;

$N=725p+218=22475t-20082$. Take $t=1$.

Ex. 67.

1. $y(3x-2)=5x+4$; or $y=\dfrac{5x+4}{3x-2}$;

$\therefore\ 3y=\dfrac{15x+12}{3x-2}=5+\dfrac{22}{3x-2}$.

The divisors of 22 are 1, 2, 11, and 22; but 1 only is here suitable; $\therefore\ 3x-2=1$; &c.

2. $x(5y+2)=3y+42$; or $x=\dfrac{3y+42}{5y+2}$;

$\therefore\ 5x=\dfrac{15y+210}{5y+2}=3+\dfrac{204}{5y+2}$.

The suitable divisors of 204 are 12, 17, 102;
thus $5y+2=12$ gives $y=2$; &c.

3. $x(3y+1)=79-y$; or $x=\dfrac{79-y}{3y+1}$;

$\therefore\ 3x=\dfrac{-3y+237}{3y+1}=-1+\dfrac{238}{3y+1}$.

The suitable divisors of 238 are 7 and 34;
thus $3y+1=7$, and $3y+1=34$.

4. $3xy+3x=4y+14$; $\therefore\ y(3x-4)=14-3x$;

$y=\dfrac{-3x+14}{3x-4}=-1+\dfrac{10}{3x-4}$.

The suitable divisors of 10 are 2 and 5;
thus $3x-4=2$, and $3x-4=5$.

5. $y(x-3)=-2x^2+2x+29$;

$\therefore\ y=\dfrac{-2x^2+2x+29}{x-3}=-2x-\dfrac{4x-29}{x-3}=-2x-4+\dfrac{17}{x-3}$.

The only suitable divisor of 17 is 1; thus $x-3=1$.

6. $y=\frac{x^2+4x+4}{x-1}=x+\frac{5x+4}{x-1}=x+5+\frac{9}{x-1}$.

Here 1, 3, 9, are all suitable values of $x-1$.

7. $x^2+x+5=9y^2-6xy+x^2$; $\therefore\ x(6y+1)=9y^2-5$;

or $x=\frac{9y^2-5}{6y+1}$; $\therefore\ 2x=\frac{18y^2-10}{6y+1}=3y-\frac{3y+10}{6y+1}$;

$\therefore\ 4x=6y-\frac{6y+20}{6y+1}=6y-1+\frac{19}{6y+1}$; hence $6y+1=19$.

8. $y=\frac{3x^2+1}{2x+1}$; $\therefore\ 2y=\frac{6x^2+2}{2x+1}=3x-\frac{3x-2}{2x+1}$;

$\therefore\ 4y=6x-\frac{6x-4}{2x+1}=6x-3+\frac{7}{2x+1}$; $\therefore\ 2x+1=7$.

Ex. 68.

1. Mult. by 12: then $3x+6+4x<6x-24+36$;

or $7x+6<6x+12$; or $7x<6x+6$; or $x<6$.

2. Let $\frac{x}{y}$ denote the fraction: then $\frac{x}{y}+\frac{y}{x}=\frac{x^2+y^2}{xy}$.

Now, $x^2+y^2>2xy$; $\therefore\ \frac{x^2+y^2}{xy}>2$.

3. (i) $\frac{a+b}{2}>\frac{2ab}{a+b}$ if $a+b>\frac{4ab}{a+b}$, or $a^2+2ab+b^2>4ab$,

or $a^2+b^2>2ab$.

(ii) $\frac{a^3+b^3}{a^2b^2}>\frac{a+b}{ab}$ if $\frac{a^2-ab+b^2}{ab}>1$, or $a^2-ab+b^2>ab$,

or $a^2+b^2>2ab$.

4. $a^6+a^4b^2+a^2b^4+b^6>a^6+2a^3b^3+b^6$,

if $a^2b^2(a^2+b^2)>a^2b^2(2ab)$.

5. By multiplication: $x^2y^2=(b^2c^2+a^2d^2)+(a^2c^2+b^2d^2)$;
but $b^2c^2+a^2d^2>2bc\,.\,ad$, or $>2ac\,.\,bd$;
$\therefore\ x^2y^2>a^2c^2+b^2d^2+2ac\,.\,bd$;
that is, $(xy)^2>(ac+bd)^2$; $\therefore\ xy>ac+bd$.
Similarly, $a^2c^2+b^2d^2>2ac\,.\,bd$, or $>2bc\,.\,ad$;
$\therefore\ x^2y^2>b^2c^2+a^2d^2+2bc\,.\,ad$; &c.

6. $\dfrac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3$; but we have $a>b$;
$\therefore\ a^2b$, ab^2, b^3, each $<a^3$; also a^3, a^2b, and ab^2, each $>b^3$;
hence $\dfrac{a^4-b^4}{a-b}<4a^3$, and $>4b^3$.

7. Since $a^2+b^2>2ab$, and $a^2+c^2>2ac$, and $b^2+c^2>2bc$,
$\therefore$ by addition, $2(a^2+b^2+c^2)>2(ab+ac+bc)$; &c.

8. Since $a^3+b^2>2ab$, $\therefore\ a^3+ab^2>2a^2b$, and $a^2b+b^3>2ab^2$;
$\therefore\ a^3+b^3+ab^2+a^2b>2ab^2+2a^2b$; $\therefore\ a^3+b^3>ab^2+a^2b$.
Similarly, $a^3+c^3>ac^2+a^2c$, and $b^3+c^3>bc^2+b^2c$;
$\therefore\ 2(a^3+b^3+c^3)>ab^2+a^2b+ac^2+a^2c+bc^2+b^2c$.

9. Since $a^2+b^2>2ab$, and $a^2+c^2>2ac$, and $b^2+c^2>2bc$;
multiply these respectively by c, b, and a, and then add.

10. $(a+b+c)^3=a^3+(b+c)^3+3a(b+c)(a+b+c)$
$=a^3+b^3+c^3+3bc(b+c)+3(ab+ac)(x+b+c)$
$=a^3+b^3+c^3+3(b^2c+bc^2+a^2b+ab^2+a^2c+ac^2)+6abc$;
$=a^3+b^3+c^3+3m+6abc$, suppose.
Now, by the 9th example, $m>6abc$, $\therefore\ 3m>18abc$;
also, by the 8th example, $a^3+b^3+c^3>\frac{1}{2}m$, that is, $>3abc$;
$\therefore\ a^3+b^3+c^3+3m+6abc>3abc+18abc+6abc$.

11. $(1-a)^4=1+6a^2+a^4-4(a+a^3)$; but $(1-a)^4$ is necessarily positive;
$\therefore\ 1+6a^2+a^4>4(a+a^3)$. Now, $1+a^2>2a$, or $a+a^3>2a^2$;
$\therefore\ 1+6a^2+a^4+(a+a^3)>4(a+a^3)+2a^2$; or
$1+4a^2+a^4>3(a+a^3)$;
$\therefore$ adding the positive quantity $(1-a^2)^2$ to the first side, we have, *à fortiori*, $2(1+a^2+a^4)>3(a+a^3)$.

12. $(a+b-c)(a+c-b)=a^2-(b-c)^2$, which is $<a^2$;
$(a+b-c)(b+c-a)=b^2-(a-c)^2$, which is $<b^2$;
$(a+c-b)(b+c-a)=c^2-(a-b)^2$, which is $<c^2$;
hence $(a+b-c)^2(a+c-b)^2(b+c-a)^2<a^2b^2c^2$.

Ex. 70.

1. $l=a+(n-1)d$, or $79=2+7(n-1)$.

2. $s=\frac{n}{2}\{2a+(n-1)d\}$, or $1200=15(2a+14\times 5)$.

3. Taking 15 times the given quantities, we have $a=196$, $d=-10$, $l=6$; then, since $l=a+(n-1)d$, we have $6=196-10(n-1)$, or $n-1=190\div 10$.

4. $s=\frac{1}{2}n\{2a+(n-1)d\}$, or $14\frac{4}{7}=\frac{1}{2}(2a+10\times\frac{2}{7})$,
or $102=11(7a+10)$, or $77a=-8$.

5. (i) $a=2$, $l=17$, $n=6$; $\therefore$ for $l=a+(n-1)d$ we have $17=2+5d$.
(ii) $a=2$, $l=-18$, $n=6$; $\therefore -18=2+5d$.

6. (i) $a=3$, $l=9$, $n=11$; $\therefore 9=3+10d$.
(ii) $a=-13$, $l=3$, $n=9$; $\therefore 3=-13+8d$.

7. (i) $a=-7$, $l=11\frac{1}{4}$, $n=12$; $\therefore 11\frac{1}{4}=-7+11d$.
(ii) $a=-3$, $l=-\frac{3}{4}$, $n=10$; $\therefore -\frac{3}{4}=-3+9d$.

8. $s=60$, $a=15$, $d=-2$: to find n.
$60=\frac{1}{2}n\{30-2(n-1)\}$, or $60=n(15-n+1)=16n-n^2$.

9. Let $x-y$, x, and $x+y$ denote the progression:
sum, $3x=21$, $\therefore x=7$; also, $4(x-y+x)=3(x+x+y)$.

10. Let $x-y$, x, and $x+y$ denote the progression;
sum, $3x=10$, $\therefore x=3\frac{1}{3}$; also, $x(x+y)=33\frac{1}{3}$,
or $3\frac{1}{3}(3\frac{1}{3}+y)=33\frac{1}{3}$, or $3\frac{1}{3}+y=10$.

11. Let $x-1$, x, $x+1$ denote the A.P.
$\therefore x(x+1)-\frac{1}{2}=x(x-1)$.

12. Here $a=n^2-n+1$, $d=2$;
$\therefore\ s=\frac{1}{2}n\{2(n^2-n+1)+2(n-1)\}=\frac{1}{2}n\times 2n^2=n^3$.

13. $425=\frac{1}{2}n\{82-2(n-1)\}$, or $n^2-42n=-425$;
whence $n=17$ or 25.
Now, $l=a+(n-1)d=41-16\times 2=9$,
or $=41-24\times 2=-7$.

14. This is to find the sum of $1+2+3+\ \ldots\ +12$;
$a=1$, $d=1$, $n=12$; hence $s=6(2+11\times 1)$.

15. Here $a=1$, $d=2$, $n=52$; $\therefore\ l=1+51\times 2=103$ shill.;
$s=26(2+51\times 2)=2704$ shill.

16. Since he will have to go to each stone and bring it back to the basket, we are to find the sum of $2+4+6+$&c. to 100 terms;
$s=50(4+99\times 2)=10100$ yds.

17. Let $x-y$, x, and $x+y$ denote the A.P.; $\therefore\ x=14$;
and $(x+y)^2-(x-y)^2=4xy=56y=616$.

18. Let $x-3y$, $x-y$, $x+y$, $x+3y$ denote the A.P.; $\therefore\ x=9$;
and $(x^2-9y^2)(x^2-y^2)=(81-9y^2)(81-y^2)=3465$;
$\therefore\ (9-y^2)(81-y^2)=385$, or $y^4-90y^2=-344$.

19. Let $x-2y$, $x-y$, x, $x+y$, $x+2y$ denote the A.P.; $\therefore\ x=\frac{1}{5}$;
sum of squares $=5x^2+10y^2=\frac{9}{40}$; or $10y^2=\frac{9}{40}-\frac{1}{5}$;

20. $l=a+(n-1)d$, or $225=a+20d$, or $a=225-20d$;
sum of 9 terms $=\frac{9}{2}(2a+8d)=9(a+4d)=9(225-16d)$;
sum of 2 terms $=2a+d=450-39d$;
$9(225-16d)=(450-39d)^2$; or $225-16d=(150-13d)^2$.

21. In the 1st series, $l=3+4(n-1)=5+6(n-1)-50$.

22. Taking $m=1, 2, 3$, successively, we have $5-2m=3, 1, -1, -3$, &c.
Sum of n terms $=\frac{1}{2}n\{6-2(n-1)\}=n(4-n)$.

23. Taking n first $=1$, and secondly $=2$, we have the 1*st* term $=\frac{1}{2}(\frac{1}{3}+\frac{1}{4})=\frac{7}{24}$, and the sum of the first two terms $=\frac{2}{3}(\frac{1}{3}+\frac{1}{4})=\frac{19}{12}$; $\therefore$ the 2*nd* term $=\frac{38-7}{24}=\frac{31}{24}$, and the comm. diff. $=\frac{31-7}{24}=1$; hence, with $a=\frac{7}{24}$, $d=1$, $n=8$, we have $l=\frac{7}{24}+7\times 1=7\frac{7}{24}$.

24. Putting $n=1, 2, 3$, successively, we have the first terms as $11-5 : 11+3$, or as $3 : 7$; the sums of the first two terms as $11-10 : 11+6$, or as $1 : 17$; and the sums of first three terms as $11-15 : 11+9$, or as $-4 : 20$, or as $-1 : 5$.

Choose, therefore, 3 and 7 as first terms, and let x and y denote the respective common differences; then the sums of the first two terms are $6+x$ and $14+y$, and those of the first three are $9+3x$ and $21+3y$, which are as $3+x : 7+y$.

Hence we have $6+x : 14+y :: 1 : 17$, or $17(6+x)=14+y$; (i)

also, $3+x : 7+y :: -1 : 5$, or $5(3+x)=-7-y$; (ii)

whence $x=-5$, $y=3$.

Accordingly, the 6th terms of the two series whose first terms are 3 and 7, and comm. differences -5 and 3, are -22 and 22, which are as $-1 : 1$.

25. Here taking a positive and a negative quantity as one term, we have to find to m terms the sum of the series $-17-33-49-$&c., which $=\frac{m}{2}\{-34-16(m-1)\}$ $=-m(8m+9)$.

26. The nth term of 1, 4, 7, &c.$=3n-2$, and of 3, 5, 7, &c. $=2n+1$; $\therefore$ the nth term of the compound series $=(3n-2)(2n+1)=6n^2-(n+2)$; hence, taking $n=1$, 2, 3, successively, the required sum will be
$S=6(1^2+2^2+3^2+\&c.)-(3+4+5+\&c.)$
$=6s-\frac{1}{2}(n^2+5n)$.

Now, to find s, the sum of $1^2+2^2+3^2+\ldots+n^2$, put A, B, C, &c. for 1, 2, 3, &c., where $B=A+1$, $C=B+1$, &c.

$B^3-A^3=(A+1)^3-A^3=3A^2+3A+1$
$C^3-B^3=(B+1)^3-B^3=3B^2+3B+1$
$D^3-C^3=(C+1)^3-C^3=3C^2+3C+1$
. . . = ... = ...
$(n+1)^3-n^3 \qquad =3n^2+3n+1.$

By addition,
$(n+1)^3-A^3=3(A^2+B^2+\&c.)+3(A+B+\&c.)+n$;
or $6s=2n^3+6n^2+6n-3n^2-3n-2n=2n^3+3n^2+n$;
whence $6s-\frac{1}{2}(n^2+5n)=S=\frac{1}{2}n(4n^2+5n-3)$.

27. The nth term of 2, 5, 8, &c. is $3n-1$, $\therefore$ that of $2^2, 5^2, 8^2$, &c. is $(3n-1)^2=9n^2-(6n-1)$. Hence, as in last example,
$S=9(1^2+2^2+3^2+\&c.)-(5+11+17+\&c.)$
$=\frac{3}{2}(2n^3+3n^2+n)-3n^2-2n=\frac{1}{2}n(6n^2+3n-1)$.

28. M's journey the 1st day$=20-2(n-1)=22-2n$;
N's do. do. $=20+5(n-1)=15+5n$;
M's whole journey$=\{44-4n+2(n-1)\}\frac{1}{2}n$
$=\frac{1}{2}n(42-2n)$;
N's whole journey$=\{30+10n-5(n-1)\}\frac{1}{2}n$
$=\frac{1}{2}n(35+5n)$;
$\therefore \frac{1}{2}n(77+3n)=343$; whence $n=7$ days.

Ex. 73.

1. $a=2$, $r=-3$, $s=-40$; $\therefore -40=2.\dfrac{(-3)^n-1}{-3-1}$;
or $20=\dfrac{(-3)^n-1}{4}$; or $(-3)^n=81$;
whence n obviously$=4$.

2. Let the series be denoted by $\frac{12}{r^2}$, $\frac{12}{r}$, 12, $12r$, $12r^2$;

$\therefore 12r^2=\frac{96}{r}$, or $r^3=8$.

3. If $4x$=the 3rd, then $16x$=the 5th, and $\sqrt{64x^2}=\pm 8x$ =the 4th; $\therefore r=\pm 2$; and the sum of 1st and 2nd, $x\pm 2x=-4$;
$\therefore x=-1\frac{1}{3}$, or 4.

4. $a=10000$, $l=14641$, $n=5$.
$10000r^4=14641$; $r^4=1{\cdot}4641$; hence $r=1{\cdot}1$, or $1\frac{1}{10}$.

5. $\frac{300}{12}=\frac{ar^3-ar^2}{ar-a}=r^2=25$; $\therefore r=\pm 5$; and $a(r-1)=12$.

6. (i) $a=2$, $l=32$, $n=5$: then $2r^4=32$.
(ii) $a=\frac{1}{2}$, $l=128$, $n=5$: then $\frac{1}{2}r^4=128$.

8. $\frac{a}{1-r}=3$, and $a+ar=2\frac{2}{3}$; $\therefore 3(1+r)(1-r)=2\frac{2}{3}$.

9. $\frac{a}{1-r}=2$, and $ar=-\frac{3}{2}$; $\therefore a=2-2r=-\frac{3}{2r}$;
hence $r^2-r=\frac{3}{4}$; $\therefore r=-\frac{1}{2}$, or $\frac{3}{2}$; and the series is 3, $-\frac{3}{2}$, $\frac{3}{4}$, &c., or -1, $-\frac{3}{2}$, $-\frac{9}{4}$, &c.

10. $\sqrt{2\frac{1}{4}\times 1}=\pm 1\frac{1}{2}$=the 2nd term; $\therefore r=\pm\frac{3}{2}\div 2\frac{1}{4}=\pm\frac{2}{3}$;
$\therefore \frac{a}{1-r}=2\frac{1}{4}\div\frac{1}{3}=6\frac{3}{4}$; or $=2\frac{1}{4}\div\frac{5}{3}=1\frac{7}{20}$.

11. If $x-y$, x, and $x+y$ denote the 2nd, 3rd, and 4th terms,
then the 1st term$=\frac{(x-y)^2}{x}$;
hence (i) $\frac{(x-y)^2(x+y)}{x}=576$; (ii) $(x-y)x=1458$;
(ii)$\div$(i) gives $\frac{x^2}{x^2-y^2}=\frac{81}{32}$; $\therefore \frac{x^2}{y^2}=\frac{81}{49}$, and $\frac{x}{y}=\pm\frac{9}{7}$;
whence $y=\pm\frac{7}{9}x$, and (ii) becomes $x(x\mp\frac{7}{9}x)=1458$.

12. The $1\frac{1}{2}$ gall. drawn each time is $\frac{1}{12}$ of the whole 18 gall.; thus the water is diminished daily in the ratio of 12 to 11; and, to find the quantity of water remaining at the end of 4 days, we have $a=2$, $r=\frac{11}{12}$, $n=5$;

$\therefore\ l=ar^{n-1}=2\times(\frac{11}{12})^4=1\frac{4273}{10368}$ gall. water,

which subtracted from 18 gall. will give the brandy required.

13. The no. of terms in A's series is 5, and in B's 4;
$\therefore\ a\times(1{\cdot}15^3-1{\cdot}1^4)=2271.$

14. (i) $x+y+z=13$; (ii) $x^2+y^2+z^2=91$; (iii) $xz=y^2$.

Squaring (i), and subtracting (ii), gives $xy+xz+yz=39$;
$\therefore\ xy+y^2+yz=39=y(x+y+z)=13y$; $\therefore\ y=3$.

From (ii): $x^2+z^2=82$; from (iii) $2xz=18$;
$\therefore\ x^2-2xz+z^2=64$, and $x-z=\pm8$; then by (i) $x+z=10$.

15. $x+xy+xy^2+xy^3$, or $x(1+y)+xy^2(1+y)$,
or $x(1+y^2)(1+y)=45$; (i)

$x^2+x^2y^2+x^2y^4+a^2y^6$, or $x^2(1+y^2)+x^2y^4(1+y^2)$,
or $x^2(1+y^4)(1+y^2)=765$; (ii)

squaring (i) and dividing by (ii) gives $\dfrac{(1+y^2)(1+y)^2}{1+y^4}=\frac{45}{17}$;

subtracting 1 from each side gives $\dfrac{2y(1+y+y^2)}{1+y^4}=\frac{28}{17}$;

$\therefore\ 28(1+y^4)=34y(1+y+y^2)$;

or $28\left(\dfrac{1}{y^2}+y^2\right)=34\left(\dfrac{1}{y}+1+y\right)$;

or $28\left(\dfrac{1}{y}+y\right)^2-56=34\left(\dfrac{1}{y}+y\right)+34.$

Assume $\dfrac{1}{y}+y=z$: then

$28z^2-34z=90$; whence $z=2\frac{1}{2}=\dfrac{1}{y}+y$; $\therefore\ y=2$ or $\frac{1}{2}$.

16. Let $x=A$'s no. of yards in the 1st minute$=B$'s in the 5th.

In A's series we have $r=1{\cdot}1$, $n=5$;

$\therefore s=\frac{x(r^n-1)}{r-1}=10x(1{\cdot}1^5-1)=6{\cdot}1051x.$

In B's series, reversed, we have $r=\frac{5}{6}$ or $1{\cdot}2$, $n=5$;

$\therefore s=5x(1{\cdot}2^5-1)=7{\cdot}4416x.$

$(7{\cdot}4416-6{\cdot}1051)x=297$; $\therefore x=222\frac{2}{3}$ yds. by A in the 1st minute$=B$'s no. in the 5th; which $\times 1{\cdot}2^4$ gives $460\frac{4}{5}$ yds. by B in the 1st minute.

Ex. 75.

3. Let $x+y$ and $x-y$ denote the nos.; then $A=x$, $G=\sqrt{x^2-y^2}$;

hence (i) $x+\sqrt{x^2-y^2}=9$; (ii) $x-\sqrt{x^2-y^2}=1$;

$\therefore$ by addition $x=5$, by subtraction $\sqrt{x^2-y^2}=4$;

we have now $x^2-y^2=16$, or $25-y^2=16$; $\therefore y=\pm 3$.

4. Let x denote the other no.; then $A=\frac{x+4}{2}$, $H=\frac{4x\times 2}{x+4}$;

hence $\frac{8x}{x+4}=\frac{16}{25}$ of $\frac{x+4}{2}$; or $\frac{x}{x+4}=\frac{x+4}{25}$;

$\therefore x^2-17x=-16.$

5. Let x and $4x$ denote the nos.; then $A=2\frac{1}{2}x$, and $H=\frac{4x^2\times 2}{x+4x}$

$=\frac{8x}{5}$;

$\therefore 2\frac{1}{2}x-1\frac{3}{5}x=1\frac{4}{5}$, or $25x-16x=18$.

6. Let x and $x+8$ denote the nos.; then $H=\frac{2x(x+8)}{2x+8}=\frac{8}{5}$;

$\therefore 5x^2+31x=36$; whence $x=1$ or $-7\frac{1}{5}$.

7. The 2nd and 3rd terms may be denoted respectively by either $x+1$ and $x-1$, or $x-1$ and $x+1$.

$x^2-1=24$; $\therefore\ x=\pm 5$;

$\therefore$ if the 2nd$=6$, the 3rd$=4$, and the 1st$=1\div(\frac{2}{3}-\frac{1}{4})=12$;
but if the 2nd$=4$, the 3rd$=6$, and the 1st$=1\div(\frac{2}{4}-\frac{1}{6})=3$.

8. (i) $x+y+z=13$; (ii) $x^2+y^2+z^2=61$; (iii) $y=\dfrac{2xz}{x+z}$.

(i) squared: $x^2+y^2+z^2+2(xy+yz)+2xz=169$;
this, by substitution from (ii) and (iii), becomes
$61+4xz+2xz=169$; or $xz=18$;

also (i), by substitution from (iii), becomes $x+z+\dfrac{2xz}{x+z}$
$=13$;

that is, $x+z+\dfrac{36}{x+z}=13$; or $(x+z)^2-13(x+z)=-36$;

whence $x+z=9$, or $x+\dfrac{18}{x}=9$, or $x^2-9x=-18$.

Ex. 76.

2. $\dfrac{a+b}{a-b}=\dfrac{a^2+2ab+b^2}{a^2-b^2}$, and $\therefore\ >\dfrac{a^2+b^2}{a^2-b^2}$.

3. (i) $\dfrac{x+y}{y}\gtrless\dfrac{4x}{x+y}$, according as $(x+y)^2\gtrless 4xy$, or as x^2+y^2
$\gtrless 2xy$. *See* (152).

(ii) $\dfrac{x^3+y^3}{x+y}\gtrless\dfrac{x^3+y^3}{x^2+y^2}$, as $x^4+y^4+2x^2y^2\gtrless x^4+y^4+xy^3+x^3y$,
as $2xy\gtrless y^2+x^2$.

(iii) $\dfrac{x^2+y^2}{1}\gtrless\dfrac{x^6+y^6}{x^4-x^3y+\&c.}$, as $1\gtrless\dfrac{x^4-2x^2y^2+y^4}{x^4-x^3y+\&c.}$,
as $-x^3y+x^2y^2-xy^3\gtrless -x^2y^2$, as $2x^2y^2\gtrless x^3y+xy^3$,
as $2xy\gtrless x^2+y^2$.

5. $\frac{a^2+ax+x^2}{a^2(a-x)+x^2(a-x)} \times \frac{a^2-ax+x^2}{a+x} = \frac{a^4+a^2x^2+x^4}{(a^2+x^2)(a-x)(a+x)}$
$=$&c.

6. $\frac{(x-5)(x-4)}{x(x-6)} \times \frac{(x-7)(x-6)}{x(x-5)} = \frac{x-4}{x} \times \frac{x-7}{x} =$&c.

7. $\frac{a+b}{a-b} \times \frac{a^2+b^2}{(a+b)^2} \times \frac{a^2-b^2}{a^2+b^2} = \frac{1}{a-b} \times \frac{1}{a+b} \times \frac{a^2-b^2}{1} = 1.$

8. $\left(\frac{a+b}{a-b}\right)^2 \times \left(\frac{a^2}{b^2}-1\right) = \frac{(a+b)^2(a+b)}{b^2(a-b)}.$

9. $\frac{a+x}{b+x} = \frac{c}{d}$; $\therefore bc+cx=ad+dx$; or $x(c-d)=ad-bc$.

10. $\frac{a-b}{a+b} \gtrless \frac{a^2-b^2}{a^2+b^2}$, as $a^3+ab^2-a^2b-b^3 \gtrless a^3-ab^2+a^2b-b^3$,
as $2ab^2 \gtrless 2a^2b$, as $b \gtrless a$.

Ex. 77.

4. $\frac{a}{b}=\frac{b}{c}$ $\therefore \frac{a^2}{b^2}=\frac{b^2}{c^2}$ $\therefore \frac{a^2+b^2}{a^2-b^2}=\frac{b^2+c^2}{b^2-c^2}=\frac{ac+c^2}{ac-c^2}=\frac{a+c}{a-c}$.

5. $\frac{a}{b}=\frac{c}{d}$ $\therefore a=\frac{bc}{d}$, and $c=\frac{ad}{b}$;

$\therefore$ (i) $(a+b)(c+d)=\left(\frac{bc}{d}+b\right)(c+d)=\frac{b(c+d)}{d}(c+d)$,
&c.

(ii) $(a+b)(c+d)=\left(\frac{ad}{b}+d\right)(a+b)=\frac{d(a+b)}{b}(a+b)$,
&c.

6. $\frac{a}{b}=\frac{c}{d}$, and $\frac{m}{n}=\frac{p}{q}$; $\therefore \frac{ma}{nb}=\frac{pc}{qd}$, and $\frac{ma+nb}{ma-nb}=$&c.

7. $\frac{a}{b}=\frac{b}{c}$ $\therefore \frac{a^2}{b^2}=\frac{b^2}{c^2}$; $\therefore \frac{a^2-b^2}{b^2}=\frac{b^2-c^2}{c^2}$; but $b^2=ac$;

$\frac{a^2-b^2}{ac}=\frac{b^2-c^2}{c^2}$; now mult. by c.

8. Since $\frac{a}{b}=\frac{e}{f}$, $\therefore \frac{a}{e}=\frac{b}{f}$, and $\frac{a-e}{e}=\frac{b-f}{f}$; $\therefore \frac{a-e}{b-f}=\frac{e}{f}=\frac{c}{d}$.

9. $\frac{a^2}{b^2}=\frac{ab}{bc}=\frac{a}{c}$; $\therefore \frac{ma^2}{nb^2}=\frac{ma}{nc}$; also $\frac{pa^2}{qb^2}=\frac{pa}{qc}$;

hence $\frac{ma^2-nb^2}{b^2}=\frac{ma-nc}{c}$, and $\frac{pa^2+qb^2}{b^2}=\frac{pa+qc}{c}$.

10. (i) Since $c=\frac{b^2}{a}$ $\therefore a-2b+c=a-2b+\frac{b^2}{a}=\frac{(a-b)^2}{a}$.

(ii) Since $a=\frac{b^2}{c}$ $\therefore a-2b+c=\frac{b^2}{c}-2b+c=\frac{(b-c)^2}{c}$.

11. If $\frac{a}{b}=\frac{c}{d}$ $\therefore d=\frac{bc}{a}$;

hence $\left(\frac{1}{a}+\frac{1}{d}\right)-\left(\frac{1}{b}+\frac{1}{c}\right)=\left(\frac{1}{a}+\frac{a}{bc}\right)-\left(\frac{1}{b}+\frac{1}{c}\right)$

$=\frac{a^2+bc-ab-ac}{abc}=\frac{a(a-b)-c(a-b)}{abc}$.

12. $\frac{a+b+c}{a-b+c}=\frac{(a+b+c)^2}{\{(a+c)+b\}\{(a+c)-b\}}=\frac{(a+b+c)^2}{(a^2+2ac+c^2)-b^2}$;

but $2ac=2b^2$; $\therefore \frac{a+b+c}{a-b+c}=\frac{(a+b+c)^2}{a^2+b^2+c^2}$.

13. (i) $\frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}=\frac{a}{b}$; $\therefore \frac{\sqrt{x}}{\sqrt{b}}=\frac{a+b}{a-b}$.

(ii) $(x+a)(4x-a)=(2x-b)(3x+b)$;

or $2x^2-(3a+b)x=b^2-a^2$;

whence $x=\frac{1}{4}(3a+b)\pm\frac{1}{4}(a+3b)$.

(iii) *First*, $\frac{x+y+2}{x+y+1}=\frac{7}{6}$; $\therefore \frac{1}{x+y+1}=\frac{1}{6}$, or $x+y=5$.

Secondly, $2y : 12y-4 :: 4x : 24x-2$,

or $1 : 6-\frac{2}{y} :: 1 : 6-\frac{1}{2x}$; $\therefore y=4x$.

(iv) *First*, $27y=9x$, or $y=\frac{1}{3}x$.
Secondly, $y(x-y)=9\times 2$; hence $\frac{1}{3}x\times\frac{8}{3}x=18$.

14. $x+1 : x+5 :: x+5 : x+13$;

$\therefore \frac{x+5}{x+1}=\frac{x+13}{x+5}$; that is, $1+\frac{4}{x+1}=1+\frac{8}{x+5}$;

$\therefore 2(x+1)=x+5$.

15. If $5x$ and $4x$ denote the nos., then

$\frac{5x-5}{4x-5}=\frac{4}{3}$, or $16x-20=15x-15$.

16. Suppose the length of the passing train to be a feet, its rate being x feet per second, and the rate at which the passenger is moving y feet per second.

In the first case, a feet is cleared at the rate of $x+y$ feet per second; which therefore occupies $\frac{a}{x+y}$ seconds $=2$.

In the second case, a feet is cleared at the rate of $x-y$ feet per second; which therefore occupies $\frac{a}{x-y}$ seconds $=30$.

$\therefore a=2(x+y)=30(x-y)$; or $7x=8y$.

17. x pounds $=A$'s stock, $y=B$'s.

(i) $x+200 : y-50 :: 4 : 1$; whence $4y-x=400$;

(ii) $x+100 : y-85 :: 60 : 13$; whence $60y-13x=6400$.

18. Suppose each leap of the hound$=3x$ inches;
$\therefore$ „ „ hare $=2x$ inches
The hare is $2x \times 50$ or $100x$ inches in advance, and leaps $8x$ inches in the time the hound leaps $9x$ inches; therefore, if the hound gains x inches in $9x$ inches, he gains $100x$ inches in $900x$ inches, that is, in 300 of his leaps.

19. (i) If A get $3x$ pounds, B gets $4x$, and C $5x$;
$\therefore$ $12x=500$; hence $3x=125$ to A; &c.
(ii) If A get $\frac{1}{3}x$, B gets $\frac{1}{4}x$, C $\frac{1}{5}x$; $\therefore$ $\frac{47}{60}x=500$;
$\therefore$ 47 : 20 :: £500 : A's share; &c.
(iii) If A's share be called 1, then B's$=\frac{3}{4}$, and C's$=\frac{3}{5}$
multiplying these proportions by $\frac{3}{2}$, we have A $1\frac{1}{2}$,
B $1\frac{1}{3}$, C $1\frac{1}{4}$.

20. $4x$ gallons milk at first become $5x$ gallons by diluting;
then $5x-3 : 5x-3+\frac{3}{4} :: 6 : 7$;
or $\frac{20x-9}{20x-12}=\frac{7}{8}$; that is, $1+\frac{3}{20x-12}=1+\frac{1}{8}$;
hence $20x-12=18$; or $20x=30$; $\therefore$ $4x=6$.

Ex. 78.

1. Since $xy \propto x^2+y^2$, $\therefore$ $xy=m(x^2+y^2)$, or $m=\frac{xy}{x^2+y^2}$
$=\frac{4\times 3}{16+9}$.

2. Let a denote the constant quantity; then $y=a+\frac{m}{x}$;
and, by the question, we have (i) $0=a+\frac{m}{2}$,
and (ii) $1=a+\frac{m}{3}$, to find the value of $a+\frac{m}{6}$.
Now, $1-\frac{m}{3}=-\frac{m}{2}$; $\therefore$ $m=-6$; also $a+\frac{m}{6}=a+\frac{m}{3}-\frac{m}{6}$
$=1-\frac{m}{6}=1+\frac{6}{6}=2$.

3. Let a be the constant; then $y=a+mxy$;

and (i) $-2\frac{1}{3}=a-4\frac{2}{3}m$, or $3a=14m-7$;
(ii) $1=a-2m$, or $3a=6m+3$;

whence $m=1\frac{1}{4}$, and $a=3\frac{1}{2}$;

$\therefore y=\frac{a}{1-mx}=\frac{3\frac{1}{2}}{1-1\frac{1}{4}x}=\frac{14}{4-5x}$.

4. $y=mx+nx^2+px^3$;

and (i) $6=m+n+p$
(ii) $22=2m+4n+8p$
(iii) $54=3m+9n+27p$
} hence $m=3$, $n=2$, $p=1$;

$\therefore y=3x+2x^2+x^3=x(x^2+2x+3)$.

5. $y=mx^2+nx+a$;

and (i) $10=m+n+a$
(ii) $17=4m+2n+a$
(iii) $26=9m+3n+a$
} hence $m=1$, $n=4$, $a=5$;

$\therefore y=x^2+4x+5$.

6. $y=mx^2$; $z=nx+ny=nx+mnx^2$;

but when $x=\frac{1}{2}$, then $y=\frac{1}{4}m=\frac{1}{3}$; $\therefore m=\frac{4}{3}$;

also $z=\frac{1}{2}n+\frac{1}{3}n=\frac{1}{4}$; $n=\frac{1}{4}\times\frac{6}{5}=\frac{3}{10}$;

$\therefore z=\frac{3}{10}x+\frac{2}{5}x^2$.

7. $x=\frac{mz}{y^2}$; and $z^2=\frac{ny}{x}$, or $x=\frac{ny}{z^2}$.

$\therefore x^2\propto\frac{z}{y^2}\times\frac{y}{z^2}$, or $x^2\propto\frac{1}{y}\times\frac{1}{z}$;

which is the case when $x\propto\frac{1}{y}$ and $x\propto\frac{1}{z}$.

8. Area$=mAC.BC$. Let $x=AC$, $y=BC$;

$x+y=14$; $x^2+y^2=100$; whence $xy=48$, or $AC.BC=48$;

$\therefore$ area$=24=\frac{1}{2}AC.BC$.

Ex. 79.

1. No. of perm. of 6 things taken 5 together, since $n-r+1=2$, is $6.5.4.3.2=720$, *1st Ans.*; which mult. by 1, for *2nd Ans.*

2. Here $7.6.5.4.3.2.1=5040$.

3. (i) *Division* has 8 letters, *i* occurring 3 times; hence
$$\frac{\lfloor 8}{1.2.3}=8.7.6.5.4=6720.$$

   (ii) *Insincere* has 9 letters, *i*, *n*, and *e* each occurring twice; hence $\dfrac{\lfloor 9}{1.2\times 1.2\times 1.2}=9.\lfloor 7=45360$.

   (iii) *Commencement* has 12 letters, *m* and *e* each occurring thrice, and *c* and *n* each twice; hence
$$\frac{\lfloor 12}{1.2.3\times 1.2.3\times 1.2\times 1.2}=11.9.8.7.600=3326400.$$

4. The expression implies 10 letters, *a* occurring 4 times, *b* thrice, *c* twice; hence
$$\frac{\lfloor 10}{1.2.3.4\times 1.2.3\times 1.2}=7.6.300=12600.$$

5. $n(n-1)(n-2) : n(n-1)(n-2)(n-3) :: 1 : n-3$;
$\therefore\ n-3=6$.

6. (i) *Mammalia* has 8 letters; *m* and *a* occurring each thrice; hence $\dfrac{\lfloor 8}{1.2.3\times 1.2.3}=8.7.20=1120$.

   (ii) *Caravansera* has 11 letters; *a* occurring 4 times, *r* twice; hence
$$\frac{\lfloor 11}{1.2.3.4\times 1.2}=11.9.7.1200=831600.$$

(iii) *Mississippi* has 11 letters; i and s each occurring 4 times, and p twice; hence

$$\frac{\lfloor 11}{1.2.3.4 \times 1.2.3.4 \times 1.2} = 34650.$$

7. $n : n(n-1)(n-2) :: 1 : n^2-3n+2=20$; whence $n=6$.

8. $n(n-1)(n-2) : (n+2)(n+1)n :: 5 : 12$;
hence $12(n^2-3n+2)=5(n^2+3n+2)$; whence $n=7$.

9. Put $n=3x$, and $\frac{2}{3}n=2x$.
$3x(3x-1)(3x-2)(3x-3) : 2x(2x-1)(2x-2)(2x-3)$
$:: 13 : 2$;
or $9(3x-1)(3x-2) : 2(2x-1)(2x-3) :: 13 : 1$;
or $9(9x^2-9x+2)=26(4x^2-8x+3)$; whence $x=5$.

10. Put $n=2x$, and $\frac{1}{2}n=x$.
$2x(2x-1)(2x-2)=12x(x-1)(x-2)$;
or $2x-1=3(x-2)$; $\therefore x=5$; $n=10$;
$\lfloor 10 = 9.8.7.12.600 = 3628800$.

Ex. 80.

1. (i) $\frac{9.8.7.6}{\lfloor 4} = 126$; (ii) Here $n-r=3$; $\therefore \frac{9.8.7}{\lfloor 3} = 84$.

(iii) Here $n-r=2$; $\therefore \frac{9.8}{1.2} = 36$.

2. (i) $\frac{11.10.9.8}{\lfloor 4} = 330$. (ii) $n-r=4$; same as (i).

(iii) $n-r=1$; $\therefore$ 11 combinations.

3. (i) No. of comb. of 10 in 15, = no. of comb. of 5 in 15,
$= \frac{15.14.13.12.11}{\lfloor 5} = 3003$.

(ii) 15 taken 12 together = 15 taken 3 together;
$\therefore \frac{15.14.13}{1.2.3} = 455$.

4. This requires to find the no. of comb. of 5 in 6, where $n-r=1$;
$\therefore$ no. of comb.$=6$.

5. *See* (190, *Cor.*): $2^6-1=63$.

6. (i) $\frac{10.9.8.7}{1.2.3.4}=210$.

(ii) Setting aside the particular person A, we have to find the no. of comb. of 3 in 9, which is $\frac{9.8.7}{1.2.3}=84$; with each of which A may now be numbered, thus making 84 comb. of 4 in 10, including A.

7. (i) $\frac{60.59.58.57.56.55}{\lfloor 6}=50063860$.

(ii) Setting aside A, as in last solution, the no. of comb. of 5 in $59=\frac{6}{60}$ or $\frac{1}{10}$ of the preceding answer.

8. $\frac{\frac{1}{2}n(\frac{1}{2}n-1)}{1.2}=15$; or $n(n-3)=270$; whence $n=18$.

9. $\frac{n(n-1)(n-2)}{1.2.3}=\frac{5}{18}$ of $\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}$;

$\therefore 3=\frac{(n-3)(n-4)}{1.2.3.4}$; or $n^2-7n+12=72$; whence $n=12$.

10. $\frac{(n+1)n(n-1)(n-2)}{1.2.3.4}=\frac{9n(n-1)}{1.2}$; or $\frac{(n+1)(n-2)}{3.4}=9$;

or $n^2-n-2=108$; whence $n=11$.

11. Put $n=6x$; then $\frac{1}{2}n=3x$, and $\frac{1}{3}n=2x$.

$\frac{3x(3x-1)(3x-2)(3x-3)}{1.2.3.4}=\frac{15}{4}$ of $\frac{2x(2x-1)(2x-2)}{1.2.3}$;

$\therefore 3(3x-1)(3x-2)=20(2x-1)$;
or $3(9x^2-9x+2)=40x-20$;
whence $x=2$; $\therefore n=6x=12$.

12. Each word is to contain 4 consonants and 2 vowels; the no. of comb. of 4 in the 21 consonants is 5985, and that of 2 in the 5 vowels is 10; there are 5985×10 different sets of the six letters; but each set may be permuted $\lfloor 6 = 720$ different ways;
$\therefore$ the no. of different words $= 59850 \times 720$.

13. $r = \frac{1}{2}n = 5$; hence $\frac{10.9.8.7.6}{\lfloor 5} = 252$.
Also, (190, *Cor.*), $2^n - 1 = 2^{10} - 1 = 1023$.

14. (i) *Holidays* has 8 different letters; $\therefore$ $r = \frac{1}{2}n = 4$.
(ii) No. of comb. of 4 in $8 = \frac{8.7.6.5}{\lfloor 4} = 70$.
Universal has 9 different letters; $\therefore$ $r = \frac{1}{2}(n-1) = 4$.
No. of comb. of 4 in $9 = \frac{9.8.7.6}{\lfloor 4} = 126$.
Hence $126 - 70 = 56$.

15. (i) $r = \frac{1}{2}n = 20 \div 2 = 10$. (ii) Now leaving out the particular man, we find the no. of comb. of 9 in 19
$$= \frac{19.18.17....11}{\lfloor 9} = 19 \times 17 \times 26 \times 11 = 92378.$$

16. The faces of A are marked 1, 2, 3,...m, and those of B 1, 2, 3,...$m+n$. Now, there are n faces of B having marks not found on A, and with these the faces of A can make mn different throws. Also the m remaining faces of B can make with the m faces of A $m(m-1)$ permutations, or $\frac{1}{2}m(m-1)$ combinations. Hence the whole no. of different throws is $mn + \frac{1}{2}m(m-1)$.

Ex. 82.

11. $\frac{1}{\sqrt[3]{(1-x)}} = (1-x)^{-\frac{1}{3}} = \&c.$

12. $\frac{1}{\sqrt{(1+x^2)}} = (1+x)^{-\frac{1}{2}} = \&c.$

Ex. 83.

1. $2^{-2}(1-\frac{1}{2}x)^{-2}=\frac{1}{4}(1-\frac{1}{2}x)^{-2}=$&c.
2. $3^{-3}(1-\frac{2}{3}x)^{-3}=\frac{1}{27}(1-\frac{2}{3}x)^{-3}=$&c.
3. $a^{-1}(1+a^{-1}bx)^{-1}=$&c.
4. $a^{-2}(1-a^{-1}b^2x)^{-2}=$&c.
5. $(a^{-\frac{1}{3}})^{-6}=a^2$; hence $a^2(1-a^{\frac{1}{3}}b^{\frac{1}{3}})^{-6}=$&c.
6. $(a^{\frac{2}{5}}(1-a^{-2}x^2)^{\frac{1}{5}}=$&c.
7. $(a^{-\frac{1}{3}})^{-3}=a$; hence $a(1+a^{\frac{1}{3}}b^{-\frac{1}{3}})^{-3}=$&c.
8. $a^{\frac{1}{3}}(1-a^{-1}x)^{\frac{1}{3}}=$&c.
9. $(a^5)^{-\frac{1}{5}}=a^{-1}$; hence $a^{-1}(1-a^{-5}x^5)^{-\frac{1}{5}}=$&c.
10. $a^{\frac{4}{3}}(1-a^{-2}x^2)^{\frac{3}{3}}=$&c.
11. $a^{-\frac{3}{2}}(1-a^{-2}x^2)^{-\frac{3}{4}}=$&c.
12. $(ax)^{\frac{1}{3}}(1-a^{-1}x)^{-\frac{1}{3}}=$&c.

Ex. 89.

1. If $a\times b^x=c$; then $x \log b+\log a=\log c$.
2. $\log a\times b^x=\log c$; $\therefore \log(\log a)+x \log b=\log(\log c)$.
3. $x \log 5=\log 800$; $\therefore x=\log 800\div\log 5$.
4. If $a^{\frac{nx}{m}}=b^{nx-a}$; then $\frac{nx}{m}\log a=(nx-a)\log b$;

   $\therefore nx \log a=(mnx-ma)\log b$;
   or $nx(m \log b-\log a)=ma \log b$.

5. Given, $x \log a+y \log b=\log c$, and $y=\frac{n}{m}x$:

   $\therefore x \log a+\frac{n}{m}x \log b=\log c$;

   or $x(m \log a+n \log b)=m \log c$.

6. $(a+b)^2(a+b)^{x-1}(a-b)^{x-1}=(a-b)^{2x}$;
or $(a+b)^{x+1}=(a-b)^{2x}\div(a-b)^{x-1}=(a-b)^{x+1}$;
$\therefore\ x+1=0$.

7. Mult. by x^x: then $x^{2x}-x^0=3x^x+3x^0$;
or $x^{2x}-1=3x^x+3$; or $x^{2x}-3x^x=4$; whence $x^x=4$, or -1;
and x manifestly $=2$, or -1.

8. $2x\ (\log 5-\log 2)+(3x-5)\log 5$
$=(x+1)(\log 5-2\log 2)+(2-x)(3\log 2-\log 5)$;
whence $3x\log 5+3x\log 2=4\log 5+5\log 2$;
$\therefore\ 3x=4$, or $x=1\frac{1}{3}$.

9. $x^{by}=y^{bx}$; that is, $x^{by}=x^{ax}$; $\therefore\ ax=by$, and $y=\frac{a}{b}x$;

hence, by (ii), $x^a=\left(\frac{ax}{b}\right)^b$, or $x^{a-b}=\left(\frac{a}{b}\right)^b$;

$\therefore\ x=\left(\frac{a}{b}\right)^{\frac{b}{a-b}}$; which mult. by $\frac{a}{b}$ to obtain y.

10. Divide by a^{2x}: then $a^{4x}-a^{2x}=1$; whence $2a^{2x}=1\pm\sqrt{5}$.
$\therefore\ \log 2+2x\log a=\log(1\pm\sqrt{5})$.

11. $a^{2x}-ba^x=-1$; whence $2a^x=b\pm\sqrt{(b^2-4)}$;
$\therefore\ \log 2+x\log a=\log(b\pm\sqrt{b^2-4})$.

12. $\frac{1}{2}(x^{\frac{1}{4}}+y^{\frac{1}{4}})\log x=\frac{4}{3}\log y$; (i)

$\frac{1}{2}(x^{\frac{1}{4}}+y^{\frac{1}{4}})\log y=\frac{1}{3}\log x$; (ii)

$\therefore\ \frac{4\log y}{\log x}=\frac{\log x}{\log y}$; or $4\log^2 y=\log^2 x$; $\therefore\ \log y=\frac{1}{2}\log x$.

From (i): $\frac{1}{2}(x^{\frac{1}{4}}+y^{\frac{1}{4}})\log x=\frac{2}{3}\log x$;

hence $\frac{1}{2}(x^{\frac{1}{4}}+y^{\frac{1}{4}})=\frac{2}{3}$;

$\therefore\ y^{\frac{2}{3}}=x^{\frac{1}{3}}$, or $y^2=x$; $\therefore\ y^{\frac{1}{4}}=x^{\frac{1}{8}}$;

$\therefore\ x^{\frac{1}{4}}+x^{\frac{1}{8}}=\frac{4}{3}$; whence $x^{\frac{1}{8}}=-\frac{1}{2}\pm\frac{1}{6}\sqrt{57}$.

Ex. 90.

1. 7) 1828
   7) 261...1
   7) 37...2
   5...2

   7) 34705
   7) 4957...6
   7) 708...1
   7) 101...1
   7) 14...3
   2...0

4. 7) 444
   7) 32...5
   2...3

   7) 4321
   7) 313...5
   7) 21...6
   1...4

5. 8) $27t$
   8) 38...1
   5...1

   8) 7007
   8) $96t$...4
   8) 122...5
   8) 17...1
   2...2

6. 4) 123
   4) 27...2
   4) 6...1
   1...2

   4) 10000
   4) 2222...1
   4) 505...0
   4) 123...2
   4) 27...2
   4) 6..1
   1...2

7. Let r denote the radix of the scale:
   $5r^2+6r+7=678$; whence $r=11$.

8. $5r^4+2r^2+6=12109$; whence $r=7$.

9. $2r^2+8r+8=289\times 2$; whence $r=15$.

10. $3r^2+5r+6=1068$; whence $r=18$.

Ex. 91.

3. 9) 1756
   9) 157...7
   9) 14...3
   1...3

   9) 345
   9) 31...4
   2...7

$(r=8)$

$$
\begin{array}{r}
1756 \\
345 \\
\hline
11646 \\
7670 \\
5712 \\
\hline
1756)\ 701746\ (345. \\
5712 \\
\hline
11054 \\
7670 \\
\hline
11646 \\
11646
\end{array}
$$

$(r=9)$

$$
\begin{array}{r}
1337 \\
274 \\
\hline
5461 \\
10584 \\
2675 \\
\hline
1337)\ 381011\ (274. \\
2675 \\
\hline
11241 \\
10584 \\
\hline
5461 \\
5461
\end{array}
$$

5. $\surd 25,40,05,44 = 4112$; $(r=6)$

$$
\begin{array}{r|l}
 & 24 \\
\hline
121 & 140 \\
 & 121 \\
\hline
1221 & 1505 \\
 & 1221 \\
\hline
12222 & 24444 \\
 & 24444 \\
\hline
\end{array}
$$

$\surd 47,61,03,70 = 6543$; $(r=9)$

$$
\begin{array}{r|l}
 & 40 \\
\hline
135 & 761 \\
 & 687 \\
\hline
1414 & 6303 \\
 & 5757 \\
\hline
14183 & 43570 \\
 & 43570 \\
\hline
\end{array}
$$

$(r=12)$

$\surd 32,e7,57,21 = 62te.$

$$
\begin{array}{r|l}
 & 30 \\
\hline
102 & 2e7 \\
 & 204 \\
\hline
104t & e357 \\
 & t404 \\
\hline
1058e & e5321 \\
 & e5321 \\
\hline
\end{array}
$$

6. The greatest no. with four figures in the scale of 6 is 5555; the least is 1000. Similarly in the other scales.

7. Let r denote the digit 12:

(r=13)		(r=6)	
6) 9008955		2013124) 4151301440 (2040.	
6) 1667r53	...0	4030252	
6) 333652	...4	12100544	
6) 70774	...4	12100544	
6) 12357	...1	10) 2040 (senary)=456 (denary).	
6) 270r	...0	10,) 113	...6
6) 568	...3	4	...5
6) $e$$r$	...1		
6) 1r	...5		
4	...1		

10. G.C.M.=34 (septenary). 11. G.C.M.=123 (octenary).

12. $\frac{6r^2+4r+7}{9r^2+5r+8}=\frac{2r+5}{3r+7}$; $\therefore \frac{18r^2+12r+21}{9r^2+5r+8}=\frac{6r+15}{3r+7}$;

or $2+\frac{2r+5}{9r^2+5r+8}=2+\frac{1}{3r+7}$;

$\therefore 9r^2+5r+8=6r^2+29r+35$; or $r^2-8r=9$;

whence $r=9$.

13. In the senary scale: $\frac{3}{12}\times\frac{41}{20}\times\frac{24}{5}=\frac{532}{2120}=2\frac{1}{2}$ (senary).

14. (i) $\cdot 04\frac{3}{4}$ (quinary)$=\frac{4\frac{3}{4}}{100}=\frac{34}{400}$ (quinary).

(ii) $\cdot 26\frac{31}{66}$ (septenary)$=\frac{26\frac{31}{66}}{100}=\frac{2602}{6600}=\frac{1301}{3300}$ (sept.)

15. $23\frac{8}{25}$ denary$=113\frac{20}{121}$ quatern.$=27\frac{10}{31}$ octen.
$=1e\frac{8}{21}$ duoden.

16. $\frac{1}{100}$ denary$=\frac{1}{1100100}$ binary$=\frac{1}{10201}$ tern.$=\frac{1}{91}$ unden.

Ex. 92.

1. $1{\cdot}05^x \times 100 = 1000$; or $1{\cdot}05^x = 10$; $\therefore x \log 1{\cdot}05 = 1$; hence $x = 1 \div {\cdot}0211893 = 47{\cdot}19$, or about 47 years.

2. The amount for 20 half-years, at $2\frac{1}{2}$ per cent. per half-year, =ditto for 20 years at $2\frac{1}{2}$ per cent. per annum, $= 1{\cdot}025^{20} \times 50$;

   $\therefore \log M = 20 \log 1{\cdot}025 + \log 50$.

   Now, $\log 1{\cdot}025 = \log 41 - \log 40 =$ ·01072387

   20

   ·21447740

   $\log 50 = 1{\cdot}6989700$

   $\log M = 1{\cdot}9134474$

   $81{\cdot}930 = 1{\cdot}9134430$

   Given tab. diff. 53) 4400

   83

   $\therefore M = £81{\cdot}93083$.

3. (i) By simp. int. $100 \div 4 = 25$ yrs.
   (ii) By comp. int. $1{\cdot}04^x = 2$; $\therefore x \log 1{\cdot}04 = \log 2$.
   Now, $\log 1{\cdot}04 = \log 26 - \log 25 = {\cdot}0173334$;
   hence $\cdot30103 \div {\cdot}017033 = 17{\cdot}6$, or nearly 18 years.

4. $M = 1{\cdot}06^{200} \div 960$. Now, $\log 1{\cdot}06 = \log 53 - \log 50$

   $= {\cdot}0253059$

   200

   $5{\cdot}0611800 = \log 115127$;

   $\therefore 115127 \div 960 = £119\ 18s.\ 6d.$, nearly.

5. By (241): $M = A \cdot \dfrac{R^n - 1}{R - 1} = £1000 \times \dfrac{1{\cdot}04^3 - 1}{{\cdot}04}$

   $= \frac{1}{4}$ of £12486·4.

6. By (240): $V = P \div (1+r)^n = £3121{\cdot}6 \div 1{\cdot}04^3 = £2775{\cdot}091$.

7. Here, (242), $V=\frac{A}{r}=\frac{A}{\cdot 04}=25A$.

8. (i) By (242): $V=\frac{A}{r}=\frac{1000}{\cdot 05}=£20000$.

(ii) Present value of £20000, due 3 years hence,
$=20000\div 1\cdot 05^3=80000\div 4\cdot 6305=£17276\cdot 75$,
nearly.

9. By (242): $V=\frac{A}{r}\cdot\frac{1}{(1+r)^m}=\frac{882}{\cdot 05}\times\frac{1}{1\cdot 05^2}=£1600$.

10. Let P pounds denote the sum borrowed. At the end of a year he has to pay $P\times 1\cdot 035$.

P pounds at $1\frac{1}{4}$ per cent. per quarter for 4 quarters, comp. int., amounts to $P\times\left(\frac{101\frac{1}{4}}{100}\right)^4=P\times(\frac{81}{80})^4$

$=P\times\frac{43046721}{80^4}$;

$\therefore\ £441=\left(\frac{43046721}{80^4}-1\cdot 035\right)P$;

or $49=\left(\frac{4782969}{80^4}-\cdot 115\right)P=\frac{72569P}{80^4}$;

$\therefore\ P=\frac{49\times 80^4}{72569}=\frac{80^4}{1481}=£27656\ 19s.\ 9d.$

11. By (242): P is equivalent to an additional rent, $Pr\div\left(1-\frac{1}{R^n}\right)$;

and the new fine (x), to a rent $xr\div\left(1-\frac{1}{R^{m+n}}\right)$, which must equal the former;

$\therefore\ \frac{xR^{m+n}}{R^{m+n}-1}=\frac{PR^n}{R^n-1}$; or $\frac{xR^m}{R^{m+n}-1}=\frac{P}{R^n-1}$.

12. Present value of A's legacy $=\frac{A}{r}\left(1-\frac{1}{E^2}\right)=\frac{25}{16}$ of B's,

$$=\tfrac{25}{16}\times\frac{A}{r}\left(\frac{1}{E^2}-\frac{1}{E^4}\right)=\tfrac{25}{16}\times\frac{A}{r}\cdot\frac{1}{E^2}\left(1-\frac{1}{E^2}\right);$$

$\therefore E^2=\frac{25}{16}$; $r=\frac{1}{4}$; hence A's legacy is worth $\frac{A}{\frac{1}{4}}(1-\frac{16}{25})$

$=£1440$; and B's is worth $\frac{16}{25}$ of $£1440=£921\ 12s.$

Ex. 94.

1. $\sqrt{5}=2+(\sqrt{5}-2)=2+\frac{1}{\sqrt{5}+2}$;

then $\frac{\sqrt{5}+2}{1}=4+\sqrt{5}-2=4+\frac{1}{\sqrt{5}+2}$;

$\therefore \sqrt{5}=2+\frac{1}{4+}\ \frac{1}{4+}\ \frac{1}{4+}$, &c.

2. $\sqrt{6}=2+(\sqrt{6}-2)=2+\frac{2}{\sqrt{6}+2}$;

$\frac{\sqrt{6}+2}{2}=2+\frac{\sqrt{6}-2}{2}=2+\frac{1}{\sqrt{6}+2}$;

$\frac{\sqrt{6}+2}{1}=4+\sqrt{6}-2=4+\frac{2}{\sqrt{6}+2}$;

$\therefore \sqrt{6}=2+\frac{1}{2+}\ \frac{1}{4+}\ \frac{1}{2+}\ \frac{1}{4+}$, &c.

3. $\sqrt{8}=2+(\sqrt{8}-2)=2+\frac{4}{\sqrt{8}+2}$;

$\frac{\sqrt{8}+2}{4}=1+\frac{\sqrt{8}-2}{4}=1+\frac{1}{\sqrt{8}+2}$;

$\frac{\sqrt{8}+2}{1}=4+\sqrt{8}-2=4+\frac{4}{\sqrt{8}+2}$;

$\therefore \sqrt{8}=2+\frac{1}{1+}\ \frac{1}{4+}\ \frac{1}{1+}\ \frac{1}{4+}$, &c.

4. $\sqrt{11}=3+(\sqrt{11}-3)=3+\dfrac{2}{\sqrt{11}+3}$;

$\dfrac{\sqrt{11}+3}{2}=3+\dfrac{\sqrt{11}-3}{2}=3+\dfrac{1}{\sqrt{11}+3}$;

$\dfrac{\sqrt{11}+3}{1}=6+\sqrt{11}-3=6+\dfrac{2}{\sqrt{11}+3}$;

$\therefore \sqrt{11}=3+\dfrac{1}{3+}\ \dfrac{1}{6+}\ \dfrac{1}{3+}\ \dfrac{1}{6+}$, &c.

5. $\sqrt{12}=3+(\sqrt{12}-3)=3+\dfrac{3}{\sqrt{12}+3}$;

$\dfrac{\sqrt{12}+3}{3}=2+\dfrac{\sqrt{12}-3}{3}=2+\dfrac{1}{\sqrt{12}+3}$;

$\dfrac{\sqrt{12}+3}{1}=6+\sqrt{12}-3=6+\dfrac{3}{\sqrt{12}+3}$;

$\therefore \sqrt{12}=3+\dfrac{1}{2+}\ \dfrac{1}{6+}\ \dfrac{1}{2+}\ \dfrac{1}{6+}$, &c.

6. $\sqrt{13}=3+(\sqrt{13}-3)=3+\dfrac{4}{\sqrt{13}+3}$;

$\dfrac{\sqrt{13}+3}{4}=1+\dfrac{\sqrt{13}-1}{4}=1+\dfrac{3}{\sqrt{13}+1}$;

$\dfrac{\sqrt{13}+1}{3}=1+\dfrac{\sqrt{13}-2}{3}=1+\dfrac{3}{\sqrt{13}+1}$;

$\dfrac{\sqrt{13}+2}{3}=1+\dfrac{\sqrt{13}-1}{3}=1+\dfrac{4}{\sqrt{13}+1}$;

$\dfrac{\sqrt{13}+1}{4}=1+\dfrac{\sqrt{13}-3}{4}=1+\dfrac{1}{\sqrt{13}+3}$;

$\dfrac{\sqrt{13}+3}{1}=6+\dfrac{\sqrt{13}-3}{1}=6+\dfrac{4}{\sqrt{13}+3}$;

$\therefore \sqrt{12}=3+\dfrac{1}{1+}\ \dfrac{1}{1+}\ \dfrac{1}{1+}\ \dfrac{1}{1+}\ \dfrac{1}{6+}\ \dfrac{1}{1+}$, &c.

7. $\sqrt{19}=4+(\sqrt{19}-4)=4+\dfrac{3}{\sqrt{19}+4}$;

$\dfrac{\sqrt{19}+4}{3}=2+\dfrac{\sqrt{19}-2}{3}=2+\dfrac{5}{\sqrt{19}+2}$;

$\dfrac{\sqrt{19}+2}{5}=1+\dfrac{\sqrt{19}-3}{5}=1+\dfrac{2}{\sqrt{19}+3}$;

$\dfrac{\sqrt{19}+3}{2}=3+\dfrac{\sqrt{19}-3}{2}=3+\dfrac{5}{\sqrt{19}+3}$;

$\dfrac{\sqrt{19}+3}{5}=1+\dfrac{\sqrt{19}-2}{5}=1+\dfrac{3}{\sqrt{19}+2}$;

$\dfrac{\sqrt{19}+2}{3}=2+\dfrac{\sqrt{19}-4}{3}=2+\dfrac{1}{\sqrt{19}+4}$;

$\therefore \sqrt{19}=4+\dfrac{1}{2+}\ \dfrac{1}{1+}\ \dfrac{1}{3+}\ \dfrac{1}{1+}\ \dfrac{1}{2+}$, &c.

8. $\sqrt{21}=4+(\sqrt{21}-4)=4+\dfrac{5}{\sqrt{21}+4}$;

$\dfrac{\sqrt{21}+4}{5}=1+\dfrac{\sqrt{21}-1}{5}=1+\dfrac{4}{\sqrt{21}+1}$;

$\dfrac{\sqrt{21}+1}{4}=1+\dfrac{\sqrt{21}-3}{4}=1+\dfrac{3}{\sqrt{21}+3}$;

$\dfrac{\sqrt{21}+3}{3}=2+\dfrac{\sqrt{21}-3}{3}=2+\dfrac{4}{\sqrt{21}+3}$;

$\dfrac{\sqrt{21}+3}{4}=1+\dfrac{\sqrt{21}-1}{4}=1+\dfrac{5}{\sqrt{21}+1}$;

$\therefore \sqrt{21}=4+\dfrac{1}{1+}\ \dfrac{1}{1+}\ \dfrac{1}{2+}\ \dfrac{1}{1+}$, &c.

12. $\sqrt{55}=7+\dfrac{6}{\sqrt{55}+7}$; then $\dfrac{\sqrt{55}+7}{6}=2+\dfrac{5}{\sqrt{55}+5}$;

$\dfrac{\sqrt{55}+5}{5}=2+\dfrac{6}{\sqrt{55}+5}$; then $\dfrac{\sqrt{55}+5}{6}=2+\dfrac{1}{\sqrt{55}+7}$;

$\dfrac{\sqrt{55}+7}{1}=14+\dfrac{6}{\sqrt{55}+7}$;

$\therefore \sqrt{55}=7+\dfrac{1}{2+}\ \dfrac{1}{2+}\ \dfrac{1}{2+}\ \dfrac{1}{14+}\ \dfrac{1}{2+}$, &c.

Ex. 95.

1. Quotients for $\frac{64}{265}$ are 4, 7, 9;

$\therefore$ its convergents are $\frac{0}{1}, \frac{1}{4}, \frac{1\times7+0}{4\times7+1}, \frac{7\times9+1}{29\times9+4}$;

or $\frac{1}{4}, \frac{7}{29}, \frac{64}{265}$.

2. Quotients for $\frac{31}{127}$ are 4, 10, 3;

$\therefore$ its convergents are $\frac{0}{1}, \frac{1}{4}, \frac{1\times10+0}{4\times10+1}, \frac{10\times3+1}{41\times3+4}$;

or $\frac{1}{4}, \frac{10}{41}, \frac{31}{127}$;

adding 2 to each gives $\frac{9}{4}, \frac{92}{41}, \frac{285}{127}$.

3. Quotients for $\frac{421}{972}$ are 2, 3, 4, 5, 6;

$\therefore$ its convergents are $\frac{0}{1}, \frac{1}{2}, \frac{3}{7}, \frac{13}{30}$, &c.

4. Quotients for $\frac{132}{611}$ are 4, 1, 1, 1, 2, 3, 1, 3;

$\therefore$ its convergents are $\frac{0}{1}, \frac{1}{4}, \frac{1}{5}, \frac{2}{9}, \frac{3}{14}$, &c.;

adding 1 to each gives $\frac{1}{1}, \frac{5}{4}, \frac{6}{5}, \frac{11}{9}, \frac{17}{14}$, &c.

7. $\sqrt{10}=3+\sqrt{10}-3=3+\frac{1}{\sqrt{10}+3}$;

$\frac{\sqrt{10}+3}{1}=6+\frac{\sqrt{10}-3}{1}=6+\frac{1}{\sqrt{10}+3}$;

thus the quotients for $\frac{1}{\sqrt{10}+3}$ are 6, 6, 6, 6, &c.;

$\therefore$ its convergents are $\frac{0}{1}, \frac{1}{6}, \frac{6}{37}, \frac{37}{228}$, &c.;

adding 3 to $\frac{6}{37}$ gives $\frac{117}{37}$.

8. $\sqrt{14}=3+\sqrt{14}-3=3+\dfrac{5}{\sqrt{14}+3}$;

$\dfrac{\sqrt{14}+3}{5}=1+\dfrac{\sqrt{14}-2}{5}=1+\dfrac{2}{\sqrt{14}+2}$;

$\dfrac{\sqrt{14}+2}{2}=2+\dfrac{\sqrt{14}-2}{2}=2+\dfrac{5}{\sqrt{14}+2}$;

$\dfrac{\sqrt{14}+2}{5}=1+\dfrac{\sqrt{14}-3}{5}=1+\dfrac{1}{\sqrt{14}+3}$;

$\dfrac{\sqrt{14}+3}{1}=6+\dfrac{\sqrt{14}-3}{1}=6+\dfrac{5}{\sqrt{14}+3}$;

thus the quotients for $\dfrac{5}{\sqrt{14}+3}$

are 1, 2, 1, 6, 1, 2, 1, 6, &c.;

$\therefore$ its convergents are $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{20}{27}$, $\frac{23}{31}$, $\frac{66}{89}$, &c.;

adding 3 to $\frac{66}{89}$ gives $\frac{333}{89}$.

9. $\sqrt{15}=3+\sqrt{15}-3=3+\dfrac{6}{\sqrt{15}+3}$;

$\dfrac{\sqrt{15}+3}{6}=1+\dfrac{\sqrt{15}-3}{6}=1+\dfrac{1}{\sqrt{15}+3}$;

$\dfrac{\sqrt{15}+3}{1}=6+\dfrac{\sqrt{15}-3}{1}=6+\dfrac{6}{\sqrt{15}+3}$;

thus the quotients for $\dfrac{6}{\sqrt{15}+3}$ are 1, 6, 1, 6, 1, 6;

$\therefore$ its convergents are $\frac{0}{1}$, $\frac{1}{1}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{48}{55}$, $\frac{55}{63}$;

10. $\sqrt{17}=4+\sqrt{17}-4=4+\dfrac{1}{\sqrt{17}+4}$;

$\dfrac{\sqrt{17}+4}{1}=8+\dfrac{\sqrt{17}-4}{1}=8+\dfrac{1}{\sqrt{17}+4}$;

thus the quotients for $\dfrac{1}{\sqrt{17}+4}$ are 8, 8, 8;

$\therefore$ its convergents are $\frac{0}{1}$, $\frac{1}{8}$, $\frac{8}{65}$;

adding 4 to $\frac{8}{65}$ gives $\frac{268}{65}$.

11. $\sqrt{18}=4+\sqrt{18}-4=4+\dfrac{2}{\sqrt{18}+4}$; &c.

The quotients for $\dfrac{2}{\sqrt{18}+4}$ will be found to be

4, 8, 4, 8, &c.;

$\therefore$ its convergents are $\frac{0}{1}, \frac{1}{4}, \frac{8}{33}, \frac{33}{136}$;

adding 4 to $\frac{8}{33}$ gives $\frac{140}{33}$.

12. $\sqrt{20}=4+\sqrt{20}-4=4+\dfrac{4}{\sqrt{20}+4}$;

the quotients for $\dfrac{4}{\sqrt{20}+4}$ will be found to be

2, 8, 2, 8, &c.

$\therefore$ its convergents are $\frac{0}{1}, \frac{1}{2}, \frac{8}{17}, \frac{17}{36}$;

adding 4 to $\frac{17}{36}$ gives $\frac{161}{36}$.

Ex. 96.

1. The quotients for $\frac{1415926}{1000000}$ are 7, 15, 1, 84, &c.

$\therefore$ its convergents are $\frac{0}{1}, \frac{1}{7}, \frac{15}{106}, \frac{16}{113}, \frac{13569}{95998}$, &c.;

hence, adding 3, we have $\frac{22}{7}, \frac{333}{106}, \frac{355}{113}$.

The approximation $\frac{22}{7}$ involves an error of *excess* $<\dfrac{1}{7\times106}$ $<\cdot0014$, and therefore may be depended on as far as *two* places of decimals. $\frac{333}{106}$ involves an error of *defect* $<\dfrac{1}{106\times113}<\cdot000084$, and therefore may be depended on to *four* places.

$\frac{355}{113}$ involves an error of *excess* $<\dfrac{1}{113\times9598}<\cdot00000092$, and therefore, although it really $=3\cdot1415929$, &c., which is correct to *six* places, it is not to be relied on beyond *five* places.

2. The quotients for $\frac{227}{685}$ are 2, 3, 3, 1;
$\therefore$ its convergents are $\frac{0}{1}$, $\frac{1}{2}$, $\frac{3}{7}$, $\frac{10}{23}$, &c.

3. $\frac{277274}{278157}=1\frac{499117}{278157}$. The quotients for the fractional part are 4, 1, 1, 1, 4, 1, 1;
$\therefore$ its convergents are $\frac{0}{1}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{2}{9}$, $\frac{3}{14}$, $\frac{14}{65}$, $\frac{17}{79}$;
adding 1 to $\frac{17}{79}$ gives $\frac{96}{79}$.

4. $\frac{479}{404}=1\frac{75}{404}$. The quotients for $\frac{75}{404}$ are 5, 2, 1, 1, &c.;
$\therefore$ its convergents are $\frac{0}{1}$, $\frac{1}{5}$, $\frac{2}{11}$, $\frac{3}{16}$, &c.;
adding 1 to $\frac{2}{11}$ gives $1\frac{3}{11}$.

5. The quotients for the fractional part are 3, 5, 7, 9, 11, &c.;
$\therefore$ its convergents are $\frac{0}{1}$, $\frac{1}{3}$, $\frac{5}{16}$, $\frac{36}{115}$, $\frac{329}{1051}$, &c.

Here $\frac{5}{16}$ involves an error of defect $< \frac{1}{16\times115} < \cdot0005$;

but $\frac{36}{115}$ involves an error of excess $< \frac{1}{115\times1051}$, that is, $< \cdot000008$, which is within $\cdot0001$; hence the value required is $1\frac{36}{115}=1\cdot3130$.

6. The diagonal of a square : the side :: $\sqrt{2} : 1$;
„ cube : the edge :: $\sqrt{3} : 1$.

(i) $\sqrt{2}=1+\sqrt{2}-1=1+\frac{1}{\sqrt{2}+1}$;

$\frac{\sqrt{2}+1}{1}=2+\frac{\sqrt{2}-1}{1}=2+\frac{1}{\sqrt{2}+1}$;

$\therefore \sqrt{2}=1+\frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{2+}$, &c.

The quotients for the fractional part are 2, 2, 2, 2, 2, 2, 2, &c.;
$\therefore$ its convergents are $\frac{0}{1}$, $\frac{1}{2}$, $\frac{2}{5}$, $\frac{5}{12}$, $\frac{12}{29}$, $\frac{29}{70}$, $\frac{70}{169}$, &c.;
adding 1 to $\frac{29}{70}$ gives $\frac{99}{70}$.

(ii) $\sqrt{3}=1+\sqrt{3}-1=1+\dfrac{2}{\sqrt{3}+1}$;

$\dfrac{\sqrt{3}+1}{2}=1+\dfrac{\sqrt{3}-1}{2}=1+\dfrac{1}{\sqrt{3}+1}$;

$\dfrac{\sqrt{3}+1}{1}=2+\dfrac{\sqrt{3}-1}{1}=2+\dfrac{2}{\sqrt{3}+1}$;

$\therefore \sqrt{3}=1+\dfrac{1}{1+}\ \dfrac{1}{2+}\ \dfrac{1}{1+}\ \dfrac{1}{2+}$, &c.

Thus the quotients for the fractional part are 1, 2, 1, 2, 1, 2, 1, 2, &c.;

$\therefore$ its convergents are $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{8}{11}$, $\frac{11}{15}$, $\frac{30}{41}$, $\frac{41}{56}$, $\frac{112}{153}$;

adding 1 to $\frac{41}{56}$ gives $\frac{97}{56}$.

Hence in (i) the limit of the error $< \dfrac{1}{70\times169}$

$=\cdot000085$;

hence in (ii) the limit of the error $< \dfrac{1}{56\times153}$

$=\cdot00012$.

7. The quotients for $\frac{65777}{100000}$ are 15, 4, 1, 8, 11, &c.;

$\therefore$ its convergents are $\frac{0}{1}$, $\frac{1}{15}$, $\frac{4}{61}$, $\frac{5}{76}$, $\frac{44}{669}$, &c.;

adding 1 gives $\frac{16}{15}$, $\frac{65}{61}$, $\frac{81}{76}$, &c.

8. (i) $x=\sqrt{\frac{7}{3}}=\dfrac{\sqrt{21}}{3}=1+\dfrac{\sqrt{21}-3}{3}=1+\dfrac{4}{\sqrt{21}+3}$.

The quotients for $\dfrac{4}{\sqrt{21}+3}$ will be found to be

$\therefore$ its convergents are $\frac{0}{1}$, $\frac{1}{1}$, $\frac{1}{2}$, $\frac{9}{17}$, $\frac{10}{19}$, $\frac{19}{36}=\cdot527$;

hence $x=1\cdot52$.

(ii) $x=\frac{5\pm\sqrt{13}}{6}$. Taking first the upper sign, we have

$$\frac{\sqrt{13}+5}{6}=1+\frac{\sqrt{13}-1}{6}=1+\frac{2}{\sqrt{13}+1}.$$

The quotients for $\frac{2}{\sqrt{13}+1}$ will be found to be

2, 3, 3, 3, &c. ;

$\therefore$ its convergents are $\frac{0}{1}, \frac{1}{2}, \frac{3}{7}, \frac{10}{23}, \frac{33}{76}=\cdot434$;

hence $x=1\cdot43$.

Taking now the lower sign: $\frac{5-\sqrt{13}}{6}=\frac{2}{\sqrt{13}+5}$,

the quotients for which will be found to be

4, 3, 3, 3, 8, &c.;

$\therefore$ its convergents are $\frac{0}{1}, \frac{1}{4}, \frac{3}{13}, \frac{10}{43}, \frac{33}{142}, \frac{109}{469}=\cdot232$;

hence $x=\cdot23$.

9. (i) $x-p=\frac{1}{q+\frac{1}{r+(x-p)}}=\frac{r+(x-p)}{qr+1+q(x-p)}$.

For $x-p$ write v:

$v=\frac{r+v}{qr+1+qv}$; or $qv^2+(qr+1)v=r+v$;

or $qv^2+qrv=r$; or $v^2+rv=\frac{r}{q}$;

whence $v=x-p=\frac{1}{2}\left\{-r\pm\sqrt{r\left(r+\frac{4}{q}\right)}\right\}$;

$\therefore\ x=p-\frac{1}{2}r\pm\frac{1}{2}\sqrt{r\left(r+\frac{4}{q}\right)}$.

(ii) Let $x=p+\frac{1}{y}$: then $y=q+\frac{1}{r+}\frac{1}{s+}\frac{1}{r+}$, &c.

$$y-q=\frac{1}{r+\dfrac{1}{s+(y-q)}}=\frac{s+(y-q)}{rs+1+r(y-q)}.$$

If for $y-q$ we write z, we shall find

$$z=y-q=\tfrac{1}{2}\left\{-s\pm\sqrt{s\left(s+\tfrac{4}{r}\right)}\right\};$$

$$\therefore\ y=q-\tfrac{1}{2}s\pm\tfrac{1}{2}\sqrt{s\left(s+\tfrac{4}{r}\right)}\Big\};$$

whence $x=p+\frac{1}{y}=p+$ &c.

10. (i) We have $x=\sqrt{(3+x)}$, or $x^2-x=3$;

whence $x=\frac{\sqrt{13}+1}{2}=2+\frac{\sqrt{13}-3}{2}=2+\frac{2}{\sqrt{13}+3}$.

The quotients for $\frac{2}{\sqrt{13}+3}$ will be found to be

3, 3, 3, 3, &c.;

$\therefore$ its convergents are $\frac{0}{1}$, $\frac{1}{3}$, $\frac{3}{10}$, $\frac{10}{33}$, $\frac{33}{109}=\cdot302$;

hence $x=2\cdot302$.

(ii) Since x must be less than 1, and negative, let

$$x=-\frac{1}{y};$$

then $\frac{1}{3^{\frac{1}{y}}}=\frac{1}{2}$; $\therefore\ 2^y=3$; and the value of y is between 1 and 2.

Put $y=1+\frac{1}{z}$; then $2\times2^{\frac{1}{z}}=3$; $\therefore\ \left(\frac{3}{2}\right)^z=2$.

The value of z is between 1 and 2.

Put $z=1+\frac{1}{w}$; then $\frac{3}{2}\left(\frac{3}{2}\right)^{\frac{1}{w}}=2$; $\therefore \left(\frac{4}{3}\right)^{w}=\frac{3}{2}$.

The value of w is between 1 and 2.

Put $w=1+\frac{1}{v}$; then $\frac{4}{3}\left(\frac{4}{3}\right)^{\frac{1}{v}}=\frac{3}{2}$; $\therefore \left(\frac{9}{8}\right)^{v}=\frac{4}{3}$.

The value of v is between 2 and 3.

Put $v=2+\frac{1}{u}$; then $\frac{81}{64}\left(\frac{9}{8}\right)^{\frac{1}{u}}=\frac{4}{3}$; $\therefore \left(\frac{256}{243}\right)^{u}=\frac{9}{8}$.

The value of u is between 2 and 3.

Thus $y=1+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}$, &c.;

$\therefore y=1\frac{7}{12}$, $x=-\frac{12}{19}=-\cdot 631$.

(iii) Here x is less than 1, and positive.

Put $x=\frac{1}{y}$; $\therefore \left(\frac{3}{2}\right)^{y}=2$; y is between 1 and 2.

Put $y=1+\frac{1}{z}$; then $\frac{3}{2}\left(\frac{3}{2}\right)^{\frac{1}{z}}=2$; $\therefore \left(\frac{4}{3}\right)^{z}=\frac{3}{2}$;

z is between 1 and 2.

Put $z=1+\frac{1}{w}$; then $\frac{4}{3}\left(\frac{4}{3}\right)^{\frac{1}{w}}=\frac{3}{2}$; $\therefore \left(\frac{9}{8}\right)^{w}=\frac{4}{3}$;

w is between 2 and 3.

Put $w=2+\frac{1}{v}$; then $\frac{81}{64}\left(\frac{9}{8}\right)^{\frac{1}{v}}=\frac{4}{3}$; $\therefore \left(\frac{256}{243}\right)^{v}=\frac{9}{8}$;

v is between 2 and 3.

Put $v=2+\frac{1}{u}$; then $\frac{65536}{59049}\left(\frac{256}{243}\right)^{\frac{1}{u}}=\frac{9}{8}$;

$\left(\frac{531441}{524288}\right)^{u}=\frac{256}{243}$; so that u is between 1 and 2.

Thus $y=1+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}$, &c.;

$\therefore y=1\frac{7}{10}$, $x=\frac{10}{17}=\cdot 58$.

Ex. 97.

5. Assume $x=Ay+By^2+\ Cy^3+$ &c.

$$\therefore\ x=Ax+Ax^2+\ Ax^3+\ Ax^4+\ Ax^5+\text{&c.}$$
$$+Bx^2+2Bx^3+3Bx^4+4Bx^5+\text{&c.}$$
$$+\ Cx^3+3Cx^4+6Cx^5+\text{&c.}$$
$$+\ Dx^4+4Dx^5+\text{&c.}$$
$$+\ Ex^5+\text{&c.}$$

Hence $A=1$; $B+A=0$, $\therefore B=-1$; $C+2B+A=0$,
$\therefore C=1$; $D+3C+3B+A=0$, $\therefore D=-1$;
$E+4D+6C+4B+A=0$, $\therefore E=1$; &c.

6. Assume as in the preceding:

$$\therefore\ x=Ax-2Ax^2+3Ax^3-\ 4Ax^4+\ 5Ax^5-\text{&c.}$$
$$+\ Bx^2-4Bx^3+10Bx^4-20Bx^5+\text{&c.}$$
$$+\ Cx^3-\ 6Cx^4+21Cx^5-\text{&c.}$$
$$+\ Dx^4-\ 8Dx^5+\text{&c.}$$
$$+\ Ex^5-\text{&c.}$$

Hence $A=1$; $B-2A=0$, $\therefore B=2$; $C-4B+3A=0$,
$\therefore C=5$; &c.

Ex. 98.

1. Here $a=3$, $n=15$, $\Delta_1=4$, $\Delta_2=3$, $\Delta_3=1$, $\Delta_4=0$;

$$\therefore\ 3+14\times4+\frac{14.13}{1.2}\times3+\frac{14.13.12}{1.2.3}\times1=696.$$

2. $a=343$, $n=11$, $\Delta_1=-6$, $\Delta_2=-5$, $\Delta_3=0$;

$$\therefore\ 343-10\times6-45\times5=58.$$

3. $a=286$, $n=7$, $\Delta_1=-81$, $\Delta_2=17$, $\Delta_3=-2$, $\Delta_4=0$;

$$\therefore\ 286-6\times81+15\times17-20\times2=15.$$

4. $a=110$, $\Delta_1=-20$, $\Delta_2=2$, $\Delta_3=0$;

hence the 11th term $=110-10\times20+45\times2=0$;
and the 12th „ $=110-11\times20+55\times2=0$.

5. $a=-10$, $n=14$, $\Delta_1=1$, $\Delta_2=1$, $\Delta_3=0$;
$\therefore\ -10+13+78=81.$

6. $a=45$, $n=10$, $\Delta_1=-30$, $\Delta_2=11$, $\Delta_3=-3$, $\Delta_4=0$;
$\therefore\ 45-9\times30+36\times11-84\times3=-81.$

7. $a=68$, $n=8$, $\Delta_1=7$, $\Delta_2=-4$, $\Delta_3=0$;
$\therefore\ 68+7\times7-21\times4=33.$

8. $a=35$, $n=12$, $\Delta_1=-33$, $\Delta_2=40$, $\Delta_3=0$;
$\therefore\ 35-11\times33+55\times40=1872.$

9. Here $a=4$, $\Delta_1=2$, $\Delta_2=3$, $\Delta_3=0$; $\therefore$ the nth term
$=4+(n-1)\times2+\frac{(n-1)(n-2)}{1.2}\times3=\frac{3n^2-5n+10}{2}$
$=226$;
whence $n=13$.

10. Here $a=20$, $\Delta_1=26$, $\Delta_2=-2$, $\Delta_3=0$; the nth term
$=20+(n-1)\times26-(n-1)(n-2)=-n^2+29n-8$
$=200$;
or $n^2-29n=-208$; whence $n=16$ or 13.

11. Here $a=168$, $\Delta_1=-31$, $\Delta_2=3$, $\Delta_3=0$; $\therefore$ the nth term
$=168-(n-1)\times31+(n^2-3n+2)\times\frac{3}{2}$
$=\frac{1}{2}(3n^2-71n+404)$, which when $=-1$ makes $n=14$,
and when $=32$ makes $n=17$.

12. $a=3$, $n=10$, $\Delta_1=2$, $\Delta_2=3$, $\Delta_3=1$, $\Delta_4=0$;
$\therefore\ s=10(3+\frac{2}{2}\times2+\frac{9.8}{2.3}\times3+\frac{9.8.7}{2.3.4}\times1)$
$=10(3+9+36+21).$

13. $a=43$, $n=12$, $\Delta_1=-16$, $\Delta_2=3$, $\Delta_3=0$;

$$\therefore\ s=12(43-\tfrac{1}{2}\times 16+\frac{11.10}{2.3}\times 3)=12(43-88+55).$$

14. Here $a=1$, $\Delta_1=3$, $\Delta_2=2$, $\Delta_3=0$;

$$\therefore\ s=n\left\{1+\frac{n-1}{2}\times 3+\frac{(n-1)(n-2)}{6}\times 2\right\}$$

$$=\frac{n}{6}(6+9n-9+2n^2-6n+4)=\frac{n}{6}(2n^2+3n+1)$$

$$=\tfrac{1}{6}n\{(2n^2+n)+(2n+1)\}=\tfrac{1}{6}n(n+1)(2n+1).$$

Hence the sum of 16 terms $=\frac{16}{6}\times 17\times 33=1496$.

15. Here $a=1$, $\Delta_1=7$, $\Delta_2=12$, $\Delta_3=6$, $\Delta_4=0$;

$$\therefore\ s=n\left\{1+\frac{n-1}{2}\times 7+\frac{(n-1)(n-2)}{2.3}\times 12+\frac{(n-1)(n-2)(n-3)}{2.3.4}\times 6\right\}$$

$$=\frac{n}{2.3.4}(24+84n-84+48n^2-144n+96+6n^3-36n^2+66n-36)$$

$$=\frac{n}{2.3.4}(6n^3+12n^2+6n)=\frac{n^2}{4}(n^2+2n+1)=\left\{\frac{n}{2}(n+1)\right\}^2.$$

16. Here $a=1$, $\Delta_1=5$, $\Delta_2=4$, $\Delta_3=0$;

$$\therefore\ s=n\left\{1+\frac{n-1}{2}\times 5+\frac{(n-1)(n-2)}{2.3}\times 4\right\}$$

$$=\frac{n}{6}(4n^2+3n-1)=\tfrac{1}{6}n\{(4n^2-n)+(4n-1)\}$$

$$=\tfrac{1}{6}n(n+1)(4n-1).$$

17. Here $a=24$, $\Delta_1=42$, $\Delta_2=18$, $\Delta_3=0$;

$$\therefore\ s=n\left\{24+\frac{n-1}{2}\times 42+\frac{(n-1)(n-2)}{2.3}\times 18\right\}$$

$$=n(3n^2+12n+9)=3n(n^2+4n+3)$$

$$=3n\{(n^2+n)+3(n+1)\}=3n(n+1)(n+3),$$

18. Here $a=1$, $\Delta_1=15$, $\Delta_2=50$, $\Delta_3=60$, $\Delta_4=24$, $\Delta_5=0$;

$$\therefore s=\left\{1+\frac{n-1}{2}\times 15+\frac{n^2-3n+2}{2.3}\times 50+\frac{n^3-6n^2+11n-6}{2.3.4}\times 60 + \frac{n^4-10n^3+35n^2-50n+24}{2.3.4.5}\times 24\right\}$$

$$=\frac{n}{120}(24n^4+60n^3+40n^2-4)=\frac{n}{30}(6n^4+15n^3+10n^2-1)$$

$$=\frac{n}{30}\{(6n^4+6n^3)+(9n^3+9n^2)+(n^2-1)\}$$

$$=\frac{n(n+1)}{30}(6n^3+9n^2+n-1).$$

Now $6n^3+9n^2+n-1=(6n^3+3n^2)+(6n^2+3n)-(2n+1)$

$=3n^2(2n+1)+3n(2n+1)-(2n+1)$

$=(2n+1)(3n^2+3n-1)$;

$\therefore s=\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$.

Ex. 98.

1. If d, the amount at 4 p.c., were given, we should have the complete series of 6 terms, having 5 orders of differences; hence we may put $\Delta_5=0$,

that is, $-(a-5b+10c-10d+5e-f)=0$;

$$\therefore d=\tfrac{1}{10}(a-5b+10c+5e-f);$$

10·240676	
112·847900	53·756655
62·118775	13·031157

$\frac{1}{10}$ of $(185{\cdot}207351-66{\cdot}787812)=11{\cdot}841954$.

[*Note.*—This Answer is the correct amount of £8 for 10 years at 4 p.c.]

2. If the value at $4\frac{1}{2}$ were given, we should have a series of 5 terms, having, therefore, 4 orders of differences;

$\therefore \Delta_4 = a - 4b + 6c - 4d + e = 0$;

whence $d = \frac{1}{4}(a - 4b + 6c + e)$;

$= \frac{1}{4}$ of $(14{\cdot}868 + 87{\cdot}12 + 14{\cdot}184 - 58{\cdot}768) = 14{\cdot}351$.

[*Note.*—This Answer is the correct value of an annuity of £4 for 4 years at $4\frac{1}{2}$ p.c.]

3. $\Delta_4 = 0 = a - 4b + 6c - 4d + e$; $\therefore c = \frac{1}{6}\{4(b+d) - (a+e)\}$
$= \frac{1}{6}$ of $15{\cdot}8129026 = 2{\cdot}6354837$.

4. $\Delta_5 = 0 = -(a - 5b + 10c - 10d + 5e - f)$;

$\therefore d = \sqrt[5]{72} = \frac{1}{10}(a - 5b + 10c + 5e - f) = 1{\cdot}23521$.

We have thus the series of 6 terms completed, and find $\Delta_1 = {\cdot}00214$, $\Delta_2 = -{\cdot}00008$, $\Delta_3 = {\cdot}00001$, $\Delta_4 = 0$.

Now as the common difference of 63, 66, &c., is 3, and 77 exceeds 75 by $\frac{2}{3}$ of 3, the 5th root of 77 is the $(5\frac{2}{3})$th term, or $n = 5\frac{2}{3}$; hence

$$\sqrt[5]{77} = 1{\cdot}22902 + 4\tfrac{2}{3} \times \Delta_1 + \frac{4\frac{2}{3} \times 3\frac{2}{3}}{2} \times \Delta_2 + \frac{4\frac{2}{3} \times 3\frac{2}{3} \times 2\frac{2}{3}}{2.3} \times \Delta_3$$

$$= 1{\cdot}22902 + {\cdot}00999 - {\cdot}00068 + {\cdot}00007 = 1{\cdot}2384.$$

5. Here $4\frac{1}{4}$ is half an interval greater than 4, and therefore we are to interpolate the $(4\frac{1}{2})$th term of the given series. We have $a = 329{\cdot}653$, $n = 4\frac{1}{2}$, $\Delta_1 = -4{\cdot}778$, $\Delta_2 = {\cdot}093$, $\Delta_3 = -{\cdot}004$, $\Delta_4 = {\cdot}002$. Hence the required value

$$= 329{\cdot}653 + 3\tfrac{1}{2} \times \Delta_1 + \frac{3\frac{1}{2} \times 2\frac{1}{2}}{2} \times \Delta_2 + \frac{3\frac{1}{2} \times 2\frac{1}{2} \times 1\frac{1}{2}}{2.3} \times \Delta_3$$

$$= 329{\cdot}653 - 16{\cdot}723 + {\cdot}407 - {\cdot}008 = 313{\cdot}329.$$

[*Note.*—This Answer is the present value of £355 due 3 years hence at $4\frac{1}{4}$ p.c.]

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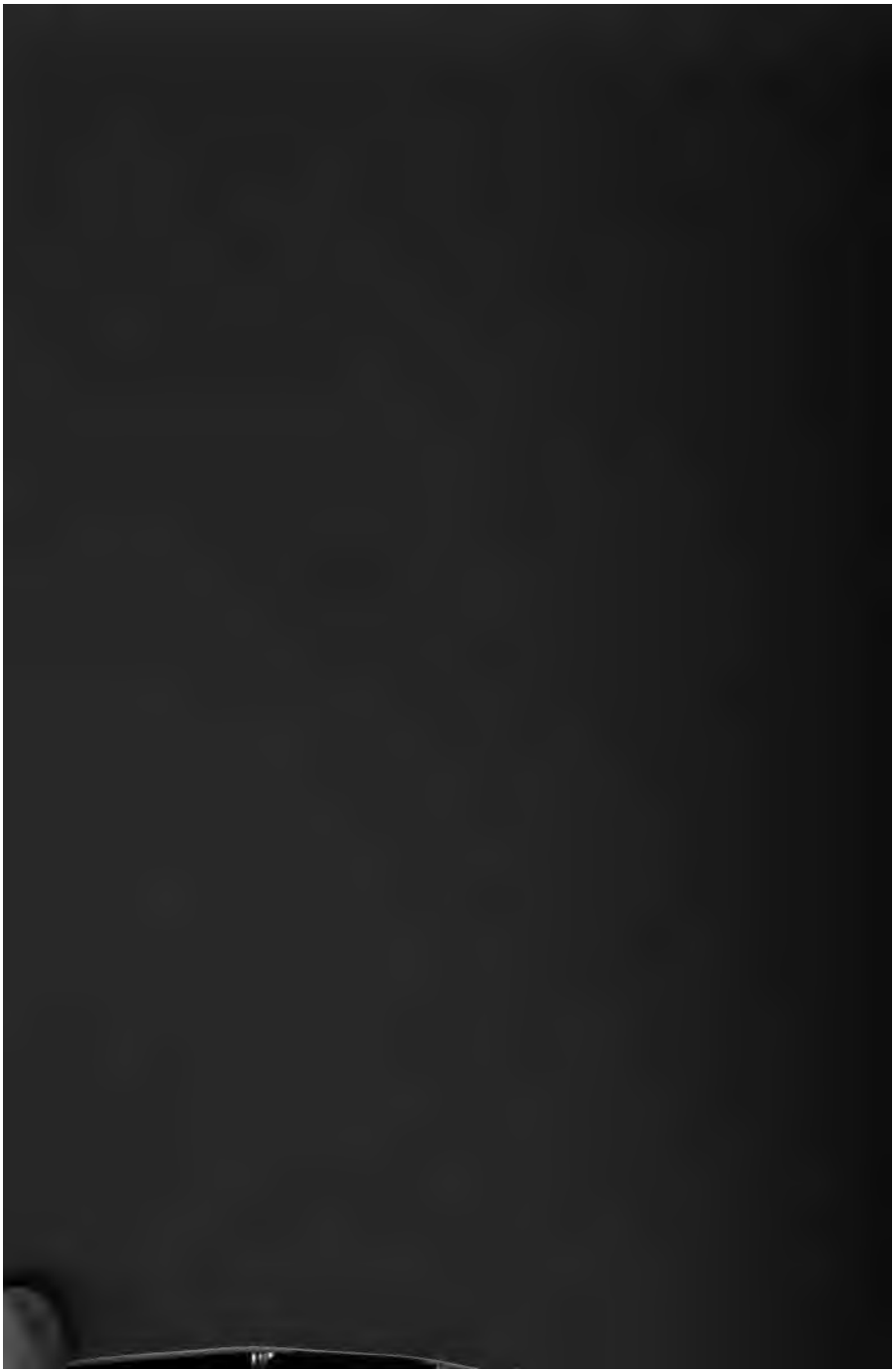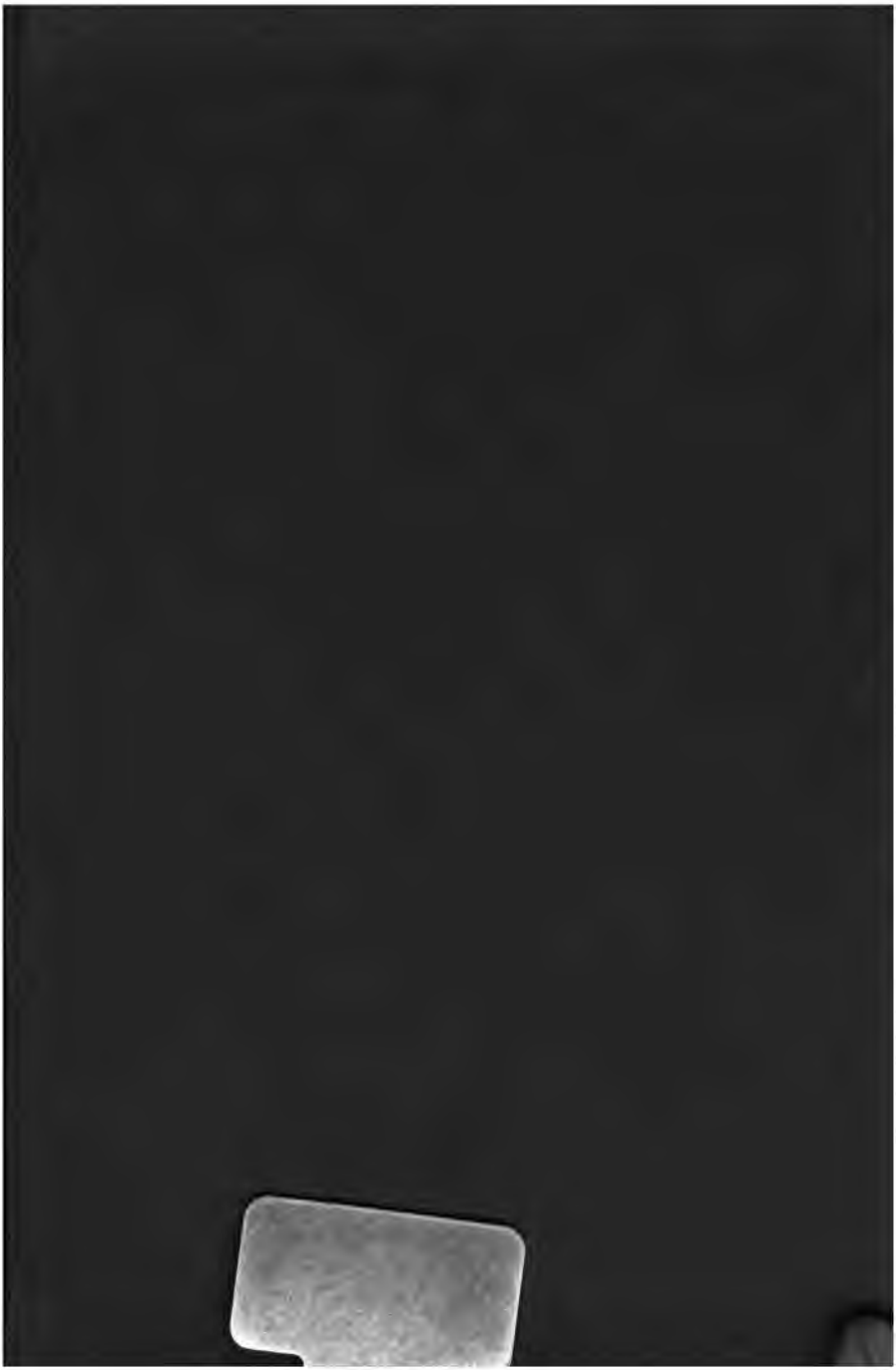
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